Distance and similarity measures between intuitionistic fuzzy sets: A comparative analysis from a pattern recognition point of view

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Abstract

A detailed analysis of the distance and similarity measures for intuitionistic fuzzy sets proposed in the past is presented in this paper. This study aims to highlight the main theoretical and computational properties of the measures under study, while the relationships between them are also investigated. Along with the literature review, a comparison of the analyzed distance and similarity measures from a pattern recognition point of view in three different classification cases is also presented. Initially, some artificial counter intuitive recognition cases are considered, while in a second phase real data from medical and well known pattern recognition benchmark problems are used to examine the discrimination abilities of the studied measures. Moreover, all the measures are applied in a face recognition problem for the first time and useful conclusions are drawn regarding the accuracy and confidence of the recognition results. Finally, the measures' suitability and their drawbacks that make the development of more robust and efficient measures' a still open issue are discussed.

Keywords: Intuitionistic fuzzy sets, distance measure, similarity measure, comparative analysis, pattern recognition, classification.

1. Introduction

Intuitionistic fuzzy sets (IFSs) have been proposed by Atanassov [1-3] as a generalization mathematical framework of the traditional fuzzy sets (FSs) originated from an early work of Zadeh [4]. The main advantage of the IFSs is their property to cope with the hesitancy that may exist due to information impression. This is achieved by incorporating a second function, along with the membership function of the conventional FSs, called non-membership function. In this way, apart from the degree of the *belongingness*, the IFSs also combine the notation of the *non-belongingness* in order to better describe the real status of the information.

Due to their property to model hesitancy and the lack of information precision, intuitionistic fuzzy sets have found application in edge detection [5], image segmentation [6,7], decision making [8], fault-tree analysis [9], pattern recognition [10,11] etc..

Among the most interesting topics in IFSs theory is the definition of appropriate measures that compare the information carried by two intuitionistic fuzzy sets. To this end, many types of measures owing different properties have been proposed in the literature such as *distance* [12-16], *similarity* [17-32], *dissimilarity* [33] measures and *entropy* [34-36], *cross-entropy* [36,37], *correlation* [38-40] and *divergence* [5,29] *indices*.

This work is focused on the distance and similarity measures between intuitionistic fuzzy sets, due to their duality (only for normalized distances) and their popularity in pattern recognition applications. The number of the proposed distance [12-16] and similarity [17-32] measures over the last twenty years is constantly increasing. This observation motivated the authors to make an exhaustive review and comparison of the most representative measures according to their importance and publication date.

Although, a short comparative analysis of some similarity measures has been conducted by Li et al. [41], there is still enough space for an additional more complete analysis with respect to some points. Compared to [41] this work extends the analysis to both distance and similarity measures and examines a wider range of both past and recent measures. Moreover, the aforementioned measures are compared to each other not only in artificial classification problems but also in real pattern recognition problems e.g. face recognition.

Summarizing, the contribution of this work is twofold. First, a significant number of the most representative distance and similarity measures are reviewed and their properties are analysed. Second, the previously analyzed measures are exhaustively compared in both artificial and real pattern recognition/classification problems. To evaluate the performance of the measures under comparison, the recognition accuracy and a *degree of confidence* index proposed herein, are used.

The paper is organized by presenting some mathematical preliminaries of IFSs in Section 2. The theoretical and computational details of the studied distance and similarity measures are introduced in Section 3 and 4, respectively. A useful review discussion of the analyzed measures is taking place in Section 5, while Section 6 evaluates their performance in pattern recognition/classification problems and summarizes the resulted experimental outcomes. Finally, some useful conclusions are drawn in Section 7.

2. Intuitionistic Fuzzy Sets (IFS)

In 1965, Zadeh proposed the theory of *fuzzy sets (FSs)* according to, a *fuzzy set (FS) A* in a universe of discourse *X* is defined as a set of ordered pairs [4],

$$A = \left\{ \left\langle x, \mu_A(x) \right\rangle / x \in X \right\},$$

where the function $\mu_A : X \to [0,1]$, define the degree of membership of the element $x \in X$.

In 1983, Atanassov [1] introduced the concept of the *intuitionistic fuzzy set (IFS)* defined as: an *intuitionistic fuzzy set A* in *X* is an object of the following form,

$$A = \left\{ \left\langle x, \mu_A(x), v_A(x) \right\rangle / x \in X \right\}$$

where the functions, $\mu_A : X \to [0,1]$ and $v_A : X \to [0,1]$, define the degree of membership and non-membership of the element $x \in X$, respectively. For every $x \in X : 0 \le \mu_A(x) + v_A(x) \le 1$ and if $\pi_A(x) = 1 - \mu_A(x) - v_A(x)$, then $\pi_A(x)$ is the hesitancy degree of the element $x \in X$ to the set *A* and $\pi_A(x) \in [0,1], \forall x \in X$.

It is easily seen that each fuzzy set is a particular case of the intuitionistic fuzzy set and in this case $\pi_A(x) = 0, \forall x \in X$.

Several relations and operations are defined [1-3], for every two intuitionistic fuzzy sets *A* and *B*, some of them are the follow:

(i) $A \subset B \Leftrightarrow \left(\mu_A(x) \le \mu_B(x) \text{ and } v_A(x) \ge v_B(x), \forall x \in X\right)$

(ii) $A \supset B \Leftrightarrow B \subset A$

(iii) $A = B \Leftrightarrow \left(\mu_A(x) = \mu_B(x) \text{ and } v_A(x) = v_B(x), \forall x \in X\right)$

(iv) $\overline{A} = \left\{\left\langle x, v_A(x), \mu_A(x)\right\rangle / x \in X\right\}$

(v) $A \cap B = \left\{\left\langle x, \min\left(\mu_A(x), \mu_B(x)\right), \max\left(v_A(x), v_B(x)\right)\right\rangle / x \in X\right\}$

(vi) $A \cup B = \left\{\left\langle x, \max\left(\mu_A(x), \mu_B(x)\right), \min\left(v_A(x), v_B(x)\right)\right\rangle / x \in X\right\}$

(vii) $A + B = \left\{\left\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x), v_A(x)v_B(x)\right\rangle / x \in X\right\}$

(viii) $A \cdot B = \left\{\left\langle x, \mu_A(x)\mu_B(x), v_A(x) + v_B(x) - v_A(x)v_B(x)\right\rangle / x \in X\right\}$

Intuitionistic fuzzy sets, proved to be a more nature information representation scheme, able to describe imprecise knowledge in respect to a specific problem. This is the reason why IFSs have attracted the scientists to develop new tools and applications to apply them.

Particular interest shows the problem of measuring the difference and/or the similarity between two IFSs, since this information is very useful in many applications of the engineering life e.g. pattern recognition.

In the following sections a detailed analysis of the most representative distance and similarity measures proposed in the last two decades and a comparative study from a pattern recognition point of view is taken place.

3. Distance measures between IFSs

Generally, a *distance* is a measure of the difference between two elements of a set. For the case of IFSs the axiomatic definitions of a distance (metric) are described in the following definition.

Definition 1: A distance (metric) d in an *intuitionistic fuzzy set A* in a universe of discourse X is a real function $d: A \times A \rightarrow R$, which satisfies the following conditions for $x, y, z \in A$:

C1. $d(x, y) \geq 0$, (non-negativity)

C2. $d(x, y) = 0 \Leftrightarrow x = y$ (coincidence)

C3. $d(x, y) = d(y, x)$ (symmetry)

C4. $d(x, z) + d(z, y) \geq d(x, y)$ (triangle inequality)

Various distance (metric) measures, involving fuzzy sets, have been proposed in [42,43]. Some of these distance measures have been extended to intuitionistic fuzzy sets, while other new distances have been introduced in the past.

The distance measures for IFSs can be categorized into two types the main members of which are analysed in the next sections. In order to simplify the distances' definitions the following notations are used.

$$\begin{aligned} \Delta_\mu(i) &= \mu_A(x_i) - \mu_B(x_i) \\ \Delta_v(i) &= v_A(x_i) - v_B(x_i) \\ \Delta_\pi(i) &= \pi_A(x_i) - \pi_B(x_i) \end{aligned} \quad (1)$$

The above notations describe the i^{th}, $i \in \{1, 2, ..., n\}$ difference between the membership, non-membership and hesitancy degree of two IFSs A, B in a universe of discourse $X = \{x_1, ..., x_n\}$.

3.1 Type I - Distance measures

From the fuzzy sets theory it is well known that if a universe set X is finite, i.e. $X = \{x_1, ..., x_n\}$ then for any two fuzzy subsets A and B of X with membership functions $\mu_A(.)$ and $\mu_B(.)$, respectively, the following distance measures d_H (*Hamming distance*), d_{nH} (normalized *Hamming distance*), d_E (*Euclidean distance*) and d_{nE} (normalized *Euclidean distance*), are defined:

$$d_H(A,B) = \sum_{i=1}^{n} \left|\Delta_\mu(i)\right| \tag{2}$$

$$d_{nH}(A,B) = \frac{1}{n}\sum_{i=1}^{n} \left|\Delta_\mu(i)\right| \tag{3}$$

$$d_E(A,B) = \sqrt{\sum_{i=1}^{n} \left(\Delta_\mu(i)\right)^2} \tag{4}$$

$$d_{nE}(A,B) = \sqrt{\frac{1}{n}\sum_{i=1}^{n} \left(\Delta_\mu(i)\right)^2} \tag{5}$$

By incorporating the non-membership function $v(.)$, Atanassov [1] has suggested a generalization of the above distances Eq.(2)-Eq.(5) for *IFSs*.

Let A, B are *IFSs* in X, with membership and non-membership functions $\mu_A(.)$, $\mu_B(.)$ and $v_A(.)$, $v_B(.)$, respectively. The above distances take the following form for the case of *IFSs*:

$$d_H^1(A,B) = \frac{1}{2}\sum_{i=1}^{n} \left[\left|\Delta_\mu(i)\right| + \left|\Delta_v(i)\right|\right] \tag{6}$$

$$d_{nH}^1(A,B) = \frac{1}{2n}\sum_{i=1}^{n} \left[\left|\Delta_\mu(i)\right| + \left|\Delta_v(i)\right|\right] \tag{7}$$

$$d_E^1(A,B) = \sqrt{\frac{1}{2}\sum_{i=1}^{n} \left[\left(\Delta_\mu(i)\right)^2 + \left(\Delta_v(i)\right)^2\right]} \tag{8}$$

$$d_{nE}^1(A,B) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n} \left[\left(\Delta_\mu(i)\right)^2 + \left(\Delta_v(i)\right)^2\right]} \tag{9}$$

On the other hand Szmidt and Kacprzyk [12] proved that the omission of the hesitant index $\pi(.)$ from the above definitions, transforms them to the corresponding distances for fuzzy sets Eq.(2)-Eq.(5) multiplied by some constant values. In the light of this observation Szmidt and Kacprzyk [12] proposed modified versions of the above distances by adding the hesitance index.

$$d_H^2(A,B)=\frac{1}{2}\sum_{i=1}^{n}\left[\left|\Delta_\mu(i)\right|+\left|\Delta_v(i)\right|+\left|\Delta_\pi(i)\right|\right] \tag{10}$$

$$d_{nH}^2(A,B)=\frac{1}{2n}\sum_{i=1}^{n}\left[\left|\Delta_\mu(i)\right|+\left|\Delta_v(i)\right|+\left|\Delta_\pi(i)\right|\right] \tag{11}$$

$$d_E^2(A,B)=\sqrt{\frac{1}{2}\sum_{i=1}^{n}\left[\left(\Delta_\mu(i)\right)^2+\left(\Delta_v(i)\right)^2+\left(\Delta_\pi(i)\right)^2\right]} \tag{12}$$

$$d_{nE}^2(A,B)=\sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left[\left(\Delta_\mu(i)\right)^2+\left(\Delta_v(i)\right)^2+\left(\Delta_\pi(i)\right)^2\right]} \tag{13}$$

On an attempt to take the advantages of the Hausdorff metric, Grzegorzewski [13] proposed Hausdorff -based distance measures, which are counterparts of those defined in Eq.(6)-Eq.(9), which after revised by Chen [44] have the form:

$$d_H^h(A,B)=\sum_{i=1}^{n}\max\left\{\left|\Delta_\mu(i)\right|,\left|\Delta_v(i)\right|\right\} \tag{14}$$

$$d_{nH}^h(A,B)=\frac{1}{n}\sum_{i=1}^{n}\max\left\{\left|\Delta_\mu(i)\right|,\left|\Delta_v(i)\right|\right\} \tag{15}$$

$$d_E^h(A,B)=\sqrt{\sum_{i=1}^{n}\max\left\{\left(\Delta_\mu(i)\right)^2,\left(\Delta_v(i)\right)^2\right\}} \tag{16}$$

$$d_{nE}^h(A,B)=\sqrt{\frac{1}{n}\sum_{i=1}^{n}\max\left\{\left(\Delta_\mu(i)\right)^2,\left(\Delta_v(i)\right)^2\right\}} \tag{17}$$

Recently, Yang and Chiclana [14] proved that the omission of the hesitant index gives different results and the incorporation of the hesitant part is in some words mandatory. Following the same procedure with Szmidt and Kacprzyk [12], they proposed a generalization of the Grzegorzewski's distances Eq.(14)-Eq.(17), which take into account the hesitant part.

$$d_H^{eh}(A,B)=\sum_{i=1}^{n}\max\left\{\left|\Delta_\mu(i)\right|,\left|\Delta_v(i)\right|,\left|\Delta_\pi(i)\right|\right\} \tag{18}$$

$$d_{nH}^{eh}(A,B)=\frac{1}{n}\sum_{i=1}^{n}\max\left\{\left|\Delta_\mu(i)\right|,\left|\Delta_v(i)\right|,\left|\Delta_\pi(i)\right|\right\} \tag{19}$$

$$d_E^{eh}(A,B)=\sqrt{\sum_{i=1}^{n}\max\left\{\left(\Delta_\mu(i)\right)^2,\left(\Delta_v(i)\right)^2,\left(\Delta_\pi(i)\right)^2\right\}} \tag{20}$$

$$d_{nE}^{eh}(A,B)=\sqrt{\frac{1}{n}\sum_{i=1}^{n}\max\left\{\left(\Delta_\mu(i)\right)^2,\left(\Delta_v(i)\right)^2,\left(\Delta_\pi(i)\right)^2\right\}} \tag{21}$$

Wang and Xin [15] aiming to resolve some unreasonable results of the previous traditional measures, proposed the following distance measures:

$$d_1(A,B)=\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\left|\Delta_\mu(i)\right|+\left|\Delta_v(i)\right|}{4}+\frac{\max\left\{\left|\Delta_\mu(i)\right|,\left|\Delta_v(i)\right|\right\}}{2}\right] \tag{22}$$

$$d_{1w}(A,B)=\sum_{i=1}^{n}w_i\left[\frac{\left|\Delta_\mu(i)\right|+\left|\Delta_v(i)\right|}{4}+\frac{\max\left\{\left|\Delta_\mu(i)\right|,\left|\Delta_v(i)\right|\right\}}{2}\right]\Bigg/\sum_{i=1}^{n}w_i \tag{23}$$

$$d_2^p(A,B)=\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}\left(\frac{\left|\Delta_\mu(i)\right|+\left|\Delta_v(i)\right|}{2}\right)^p} \tag{24}$$

In the above formulas, w_i corresponds to the weight of the $x_i \in X=\{x_1,x_2,...,x_n\}$ element, with $0 \le w_i \le 1$ and p is a positive integer. It is obvious that the d_1 distance is a special case of the d_{1w}, which is derived from Eq.(23) for the weight set $w_i = 1/n,\ i \in \{1,2,...,n\}$.

3.2 Type II - Distance measures

The aforementioned distance measures are characterized by the linear or non-linear relationship of the membership and non-membership functions $\mu(.)$ and $v(.)$ of the IFSs, respectively. However, the authors have proposed [16] a different type of distance measures based on *fuzzy implications*.

A *fuzzy implication* is a function $\sigma_{\Rightarrow}:[0,1]\times[0,1]\rightarrow[0,1]$, which for any truth values a, $b\in[0,1]$ of (fuzzy) propositions p, q, respectively, gives the truth value $\sigma_{\Rightarrow}(a,b)$, of conditional proposition " if p then q ". Function $\sigma_{\Rightarrow}(.,.)$ should be an extension of the *classical implication* from the domain $\{0,1\}$ to the domain $[0,1]$.

The *implication operator* of classical logic is a map $m:\{0,1\}\times\{0,1\}\rightarrow\{0,1\}$ which satisfies the following conditions: $m(0,0)=m(0,1)=m(1,1)=1$ and $m(1,0)=0$. The latter conditions are typically the minimum requirements for a fuzzy implication operator. In other words, fuzzy implications are required to reduce to the classical implication when truth-values are restricted to 0 and 1; i.e. $\sigma_{\Rightarrow}(0,0)=\sigma_{\Rightarrow}(0,1)=$ $\sigma_{\Rightarrow}(1,1)=1$ and $\sigma_{\Rightarrow}(1,0)=0$.

A number of basic properties of the classical (logic) implication have been generalized by fuzzy implications. Hence, a number of "reasonable axioms" emerged tentatively for fuzzy implications. Some of the aforementioned axioms are displayed next [45].

A1. $a\le b\Rightarrow\sigma_{\Rightarrow}(a,x)\ge\sigma_{\Rightarrow}(b,x)$	*Monotonicity in the first argument*
A2. $a\le b\Rightarrow\sigma_{\Rightarrow}(x,a)\le\sigma_{\Rightarrow}(x,b)$	*Monotonicity in the second argument*
A3. $\sigma_{\Rightarrow}(a,\sigma_{\Rightarrow}(b,x))=\sigma_{\Rightarrow}(b,\sigma_{\Rightarrow}(a,x))$	*Exchange property*
A4. $\sigma_{\Rightarrow}(a,b)=\sigma_{\Rightarrow}(n(b),n(a))$	*Contraposition*
A5. $\sigma_{\Rightarrow}(1,b)=b$	*Neutrality of truth*
A6. $\sigma_{\Rightarrow}(0,a)=1$	*Dominance of falsity*
A7. $\sigma_{\Rightarrow}(a,a)=1$	*Identity*
A8. $\sigma_{\Rightarrow}(a,b)=1\Leftrightarrow a\le b$	*Boundary condition*
A9. $\sigma_{\Rightarrow}$ is a continuous function	*Continuity*

The properties of some representative fuzzy implications [45] are summarized in the following Table 1.:

Table 1. Properties of some fuzzy implications.

Implication Name	Implication Definition	Description
Reichenbach	$\sigma_{\mathrm{R}}(a,b)=1-a+ab$	*S*-implication
Gödel	$\sigma_{\mathrm{G}}(a,b)=\begin{cases}1, \text{ for } a\le b\\ b, \text{ for } a>b\end{cases}$	*R*-implication
Lukasiewicz	$\sigma_{\mathrm{L}}(a,b)=\min\{1,1-a+b\}$	*S- and R*-implication
Kleene-Dienes	$\sigma_{\mathrm{KD}}(a,b)=\max\{1-a,b\}$	*S*- and *QL*-implication
Mamdani	$\sigma_{\mathrm{M}}(a,b)=\min\{a,b\}$	engineering implication
Larsen	$\sigma_{\mathrm{La}}(a,b)=ab$	engineering implication

Moreover, a novel fuzzy implication $\sigma_T(a,b)=\dfrac{f(b)}{f(a\vee b)}$ has been presented in [46], where $a,b\in[0,1]$ and $f:[0,1]\to[0,1]$, stemming from a fuzzy lattice inclusion measure function. It was shown that the aforementioned fuzzy implication satisfies a number of "reasonable axioms" and properties of fuzzy implications.

Following the same process with [47], a new family of metric distances between intuitionistic fuzzy sets based on matrix norms and fuzzy implications are derived.

Furthermore, it is remarked [48] that if $\Pi_1=(a_{ij}),\Pi_2=(b_{ij}),i=1,\ldots,n,\ j=1,\ldots,n$ are square matrices then the norm $\|\ \|$ can be used to define a metric d as:

$$d(\Pi_1,\Pi_2)=\|\Pi_1-\Pi_2\| \tag{25}$$

Let A be *IFS* in a finite universe $X=\{x_1,...,x_n\}$, with membership functions $\mu_A(.)$, and with non-membership functions $v_A(.)$, respectively. Let $\sigma_{\Rightarrow}$ be a fuzzy implication. We define the $n\times n$ matrices $\Pi(\mu_A)$ and $\Pi(v_A)$ of $\sigma_{\Rightarrow}$ as follows:

$$\Pi(\mu_A)\triangleq\left[\underset{i=1,\ldots,n}{\sigma_{\Rightarrow}}\left(\mu_A(x_i),\mu_A(x_i)\right)\right]=\sigma_{\Rightarrow}\left(\begin{bmatrix}\mu_A(x_1)\\ \vdots\\ \mu_A(x_n)\end{bmatrix},\left[\mu_A(x_1),\ldots,\mu_A(x_n)\right]\right)$$

$$=\begin{bmatrix}\sigma_{\Rightarrow}(\mu_A(x_1),\mu_A(x_1)) & \cdots & \cdots & \sigma_{\Rightarrow}(\mu_A(x_1),\mu_A(x_n))\\ \vdots & \vdots & \vdots & \vdots\\ \sigma_{\Rightarrow}(\mu_A(x_n),\mu_A(x_1)) & \cdots & \cdots & \sigma_{\Rightarrow}(\mu_A(x_n),\mu_A(x_n))\end{bmatrix} \quad \text{and}$$

$$\Pi(v_A) \triangleq \left[\underset{i=1,\ldots,n}{\sigma_{\Rightarrow}} \left(v_A(x_i), v_A(x_i) \right) \right] = \sigma_{\Rightarrow}\left(\begin{bmatrix} v_A(x_1) \\ \vdots \\ v_A(x_n) \end{bmatrix}, \left[v_A(x_1), \ldots, v_A(x_n) \right] \right)$$

$$= \begin{bmatrix} \sigma_{\Rightarrow}\left(v_A(x_1), v_A(x_1)\right) & \cdots & \cdots & \sigma_{\Rightarrow}\left(v_A(x_1), v_A(x_n)\right) \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{\Rightarrow}\left(v_A(x_n), v_A(x_1)\right) & \cdots & \cdots & \sigma_{\Rightarrow}\left(v_A(x_n), v_A(x_n)\right) \end{bmatrix}, \text{ respectively.}$$

Let X denote a universe of discourse, where X is a finite and let Σ_{IFSs}^{X} denote the set of all $IFSs$ in X.

Definition 2: Given two intuitionistic fuzzy sets, $A = \{\langle x, \mu_A(x), v_A(x)\rangle / x \in X\}$ and $B = \{\langle x, \mu_B(x), v_B(x)\rangle / x \in X\}$, where $X = \{x_1, \ldots, x_n\}$ is a finite universe of discourse. Also, let $\sigma_{\Rightarrow}$ be a fuzzy implication and any tensor-or operator-norm $\|\cdot\|$. Then

$$d(A,B;\sigma_{\Rightarrow}) \triangleq \left\| \Pi(\mu_A) - \Pi(\mu_B) \right\| + \left\| \Pi(v_A) - \Pi(v_B) \right\| \tag{26}$$

where $\Pi(\mu_\cdot) = \left[\underset{i=1,\ldots,n}{\sigma_{\Rightarrow}} \left(\mu_\cdot(x_i), \mu_\cdot(x_i) \right) \right]$, $\Pi(v_\cdot) = \left[\underset{i=1,\ldots,n}{\sigma_{\Rightarrow}} \left(v_\cdot(x_i), v_\cdot(x_i) \right) \right]$, defines a metric distance $d : \Sigma_{IFSs}^{X} \times \Sigma_{IFSs}^{X} \to [0, +\infty)$.

The above function $d(A,B;\sigma_{\Rightarrow})$ is a metric [47]. So, this definition actually introduces multiple distance metrics with different meanings, according to the used fuzzy implication.

In Eq.(26) the norm $\|\Pi\|$ is computed by using the largest non negative eigenvalue of the positive definite Hermitian matrix $\Pi^T \Pi$ (Π^T is the transpose of matrix Π) [48],

$$\|\Pi\| = \sqrt{\lambda_{\max}} \tag{26}$$

It is worth noting that the main advantage of the fuzzy implication based distance measure Eq.(26) is its flexibility, which permits different fuzzy implications to be incorporated by extending its applicability to several applications where the most appropriate implication is used.

The distance measures obtained by using the fuzzy implications of Table 1 and the $\sigma_T\left(a,b\right)$ one are hereafter used in a large scale comparative analysis. More precisely, the distance measures d_R^{imp}, d_G^{imp}, d_L^{imp}, d_{KD}^{imp}, d_M^{imp}, d_{La}^{imp} and d_T^{imp} defined in Eq.(26) when the fuzzy implications σ_R, σ_G, σ_L, σ_{KD}, σ_M, σ_{La} and σ_T defined in Table 1, are used respectively.

The authors have extended the above distance measure to D-implications [49] for fuzzy sets and can be generalized herein for the case of the intuitionistic fuzzy sets. Therefore, in the following analysis an additional distance metric (d^{Dimp}) based on the D-implication $I_D\left(a,b\right)=\max\left\{\min\left\{1-a,1-b\right\},b\right\}$ is formed by using Eq.(26).

4. Similarity measures between IFSs

By revising the definition of a symmetry measure proposed by Dengfeng and Chuntian [20], Mitchell introduced the definition of the *strong* symmetry measure [22], which is more appropriate for pattern recognition applications and defined as:

Definition 3: A similarity measure S in an *intuitionistic fuzzy set A* in a universe of discourse *X* is a real function $S: A\times A\rightarrow R$, which satisfies the following conditions for $x,y,z\in A$:

S1. $0\leq S\left(x,y\right)\leq 1$, (non-negativity)

S2. $S\left(x,y\right)=1 \textit{ if and only if } x=y$ (coincidence)

S3. $S\left(x,y\right)=S\left(y,x\right)$ (symmetry)

S4. if $x\subseteq y\subseteq z$, then $d\left(x,z\right)\leq d\left(x,y\right)$ (triangle inequality)

$\textit{and } d\left(x,z\right)\leq d\left(y,z\right)$

Considering the outcome of the analysis presented by Bustine and Burillo [50], which concluded that the intuitionistic fuzzy sets and the vague sets are similar, Chen [17] proposed the first similarity measure for IFSs defined as:

$$S_C^1(A,B)=\frac{1}{n}\sum_{i=1}^{n}\left(1-\left|\frac{S_A(x_i)-S_B(x_i)}{2}\right|\right) \tag{27}$$

$$S_C^{1w}(A,B)=\sum_{i=1}^{n}w_i\left(1-\left|\frac{S_A(x_i)-S_B(x_i)}{2}\right|\right)\Bigg/\sum_{i=1}^{n}w_i \tag{28}$$

where $S_k(x_i)=\mu_k(x_i)-\nu_k(x_i)$, $k=\{A,B\}$ and $0\le w_i\le 1$. It is obvious that S_C^1 is a special case of S_C^{1w} and is derived from Eq.(28) for the weight set $w_i=1/n,\ i\in\{1,2,...,n\}$.

Furthermore, Chen [18] proposed a weighted similarity measure between vague sets

$$S_C^w(A,B)=\sum_{i=1}^{n}w_i\left(1-\frac{\left|a*\Delta_\mu(i)+b*\Delta_\nu(i)-c*\left(\Delta_\mu(i)+\Delta_\nu(i)\right)\right|}{a-b}\right)\Bigg/\sum_{i=1}^{n}w_i \tag{29}$$

where $0\le w_i\le 1$, $i\in\{1,2,...,n\}$ and $a\ge c\ge 0\ge b$.

Hong and Kim [19] found that Chen's similarity measures, shown unreasonable results in some cases and proposed the following measures:

$$S_H^1(A,B)=\frac{1}{n}\sum_{i=1}^{n}\left(1-\frac{\left|\Delta_\mu(i)\right|+\left|\Delta_\nu(i)\right|}{2}\right) \tag{30}$$

$$S_H^w(A,B)=\sum_{i=1}^{n}w_i\left(1-\frac{a*\left|\Delta_\mu(i)\right|+b*\left|\Delta_\nu(i)\right|+c*\left|\Delta_\mu(i)+\Delta_\nu(i)\right|}{a+b+c}\right) \tag{31}$$

where $0\le w_i$, $i\in\{1,2,...,n\}$ and $a,b,c\ge 0$.

The years to follow the latest work of Hong and Kim [19], the beginning of a tremendous development of many similarity measures was pointed out. The first work of this period was from Denfeng and Chuntian [20] who proposed two new similarity measures.

$$S_d^p(A,B)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}\left|\varphi_A(i)-\varphi_B(i)\right|^p} \tag{32}$$

$$S_{dw}^p(A,B)=1-\sqrt[p]{\sum_{i=1}^{n}w_i\left|\varphi_A(i)-\varphi_B(i)\right|^p} \tag{33}$$

where $\varphi_k(i)=\left(\mu_k(x_i)+1-\nu_k(x_i)\right)/2$, $k=\{A,B\}$, $1\le p<+\infty$, $0\le w_i\le 1$ $i\in\{1,2,...,n\}$ and $\sum_{i=1}^{n}w_i=1$. Similarity measure S_d^p is a special case of S_{dw}^p and is derived from Eq.(33) for the weight set $w_i=1/n,\ i\in\{1,2,...,n\}$.

Liang and Shi [21], proposed the following four similarity measures, claiming that they contain more information in IFSs than those of Denfeng and Chuntian [20].

$$S_e^p(A,B)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}\left(\frac{\left|\Delta_\mu(i)\right|+\left|\Delta_\nu(i)\right|}{2}\right)^p} \tag{34}$$

$$S_s^p(A,B)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}\left(\varphi_{s1}(i)+\varphi_{s2}(i)\right)^p} \tag{35}$$

where

$\varphi_{s1}(i)=\left|m_{A1}(x_i)-m_{B1}(x_i)\right|/2$, $\varphi_{s2}(i)=\left|m_{A2}(x_i)-m_{B2}(x_i)\right|/2$, and

$m_{k1}(x_i)=\left(\mu_k(x_i)+m_k(x_i)\right)/2$, $m_{k2}(x_i)=\left(m_k(x_i)+1-\nu_k(x_i)\right)/2$,

$m_k(x_i)=\left(\mu_k(x_i)+1-\nu_k(x_i)\right)/2$, $k=\{A,B\}$.

$$S_h^p(A,B)=1-\frac{1}{\sqrt[p]{n}}\sqrt[p]{\sum_{i=1}^{n}\left(\sum_{m=1}^{3}\omega_m\varphi_m(i)\right)^p} \tag{36}$$

where $0\le\omega_m\le 1$, $\sum_{i=1}^{3}\omega_i=1$,

$\varphi_1(i)=\left(\left|\Delta_\mu(i)\right|+\left|\Delta_\nu(i)\right|\right)/2$, $\varphi_2(i)=\left|m_A(i)-m_B(i)\right|$,

$\varphi_3(i)=\max\left(l_A(i),l_B(i)\right)-\min\left(l_A(i),l_B(i)\right)$ with $l_k(i)=m_k(i)-\mu_k(i), k=\{A,B\}$.

$$S_w^p(A,B) = 1 - \sqrt[p]{\sum_{i=1}^{n} w_i \left(\sum_{m=1}^{3} \omega_m \varphi_m(i) \right)^p} \tag{37}$$

with $0 \le w_i \le 1 \;\; i \in \{1,2,...,n\}$ and $\sum_{i=1}^{n} w_i = 1$. It is obvious that S_h^p is a special case of S_w^p and is derived from Eq.(37) for the weight set $w_i = 1/n,\, i \in \{1,2,...,n\}$.

Park et al. [22], showed that although the similarity measure S_e^p gives reasonable results in most cases, there are other ones where this measure does not work properly. In order to overcome this deficiency of S_e^p Park et al. [22] proposed the following alternative similarity measures:

$$S_g^p(A,B) = 1 - \frac{1}{\sqrt[p]{n}} \sqrt[p]{\sum_{i=1}^{n} \left(\frac{|\Delta_\mu(i)| + |\Delta_\nu(i)| + |\Delta_\pi(i)|}{2} \right)^p} \tag{38}$$

$$S_{gw}^p(A,B) = 1 - \sqrt[p]{\sum_{i=1}^{n} w_i \left(\frac{|\Delta_\mu(i)| + |\Delta_\nu(i)| + |\Delta_\pi(i)|}{2} \right)^p} \tag{39}$$

with $1 \le p < +\infty$, $0 \le w_i \le 1 \;\; i \in \{1,2,...,n\}$ and $\sum_{i=1}^{n} w_i = 1$. It is obvious that S_g^p is a special case of S_{gw}^p and is derived from Eq.(39) for the weight set $w_i = 1/n,\, i \in \{1,2,...,n\}$.

Mitchell [23] pointed out that the similarity measures of Denfeng and Chuntian [20] give some counter-intuitive results, by characterizing two different IFSs as identical. Mitchell tried to overcome this drawback by revising the definition of a similarity measure and by providing a more realistic strong similarity measure of the following form:

$$S_{\text{mod},p}(A,B) = \frac{1}{2}\left(\rho_\mu(A,B) + \rho_f(A,B) \right) \tag{40}$$

where $\rho_\mu(A,B) = S_d^p\left(\mu_A(x_i), \mu_B(x_i)\right)$ and $\rho_f(A,B) = S_d^p\left(1 - \nu_A(x_i), 1 - \nu_B(x_i)\right)$.

Recently, Julian et al. [24] have questioned the way the experimental results of Mitchell [22] are derived and proposed a more consistent to those results similarity measure defined as:

$$S_{new,p}\left(A,B\right)=1-\sqrt[p]{\sum_{i=1}^{n}w_i\left(\left|\Delta_\mu\left(i\right)\right|\right)^p}-\sqrt[p]{\sum_{i=1}^{n}w_i\left(\left|\Delta_v\left(i\right)\right|\right)^p} \tag{41}$$

with $w_i \geq 0 \;\; i\in\{1,2,...,n\}$ and $\sum_{i=1}^{n}w_i=1$ and $p\geq 1$.

However, by analysing Julian's similarity Eq.(41), Tung et al. [51] found that this similarity does not satisfy condition (S1) concerning the non-negativity condition of Definition 3.

Working in the same way as Grzegorzewski [13], Hung and Yang [25] introduced a new set of similarity measures based on the Hausdorff distance $d_H\left(A,B\right)=\frac{1}{n}\sum_{i=1}^{n}\max\left\{\left|\Delta_\mu\left(i\right)\right|,\left|\Delta_v\left(i\right)\right|\right\}$.

$$S_l\left(A,B\right)=1-d_H\left(A,B\right) \tag{42}$$

$$S_e\left(A,B\right)=\frac{e^{-d_H\left(A,B\right)}-e^{-1}}{1-e^{-1}} \tag{43}$$

$$S_c\left(A,B\right)=\frac{1-d_H\left(A,B\right)}{1+d_H\left(A,B\right)} \tag{44}$$

Moreover, if the weighted Hausdorff distance is used then the weighted versions of the above similarities are derived.

In an attempt to resolve some unreasonable results presented by the similaritiy measures of Hong and Kim [19] and Denfeng and Chuntian [20], and working in the same way as Szmidt and Kacprzyk [12], Liu [26] proposed the following similarity measures:

$$S_L^p\left(A,B\right)=1-\sqrt[p]{\frac{1}{2n}\sum_{i=1}^{n}\left[\left|\Delta_\mu\left(i\right)\right|^p+\left|\Delta_v\left(i\right)\right|^p+\left|\Delta_\pi\left(i\right)\right|^p\right]} \tag{45}$$

$$S_{Lw}^{p}\left(A,B\right)=1-\sqrt[p]{\sum_{i=1}^{n}w_i\left[a*\left|\Delta_{\mu}\left(i\right)\right|^{p}+b*\left|\Delta_{v}\left(i\right)\right|^{p}+c*\left|\Delta_{\pi}\left(i\right)\right|^{p}\right]} \quad (46)$$

where $1<p<+\infty$ $0\le w_i\le 1$ $i\in\{1,2,...,n\}$, $\sum_{i=1}^{n}w_i=1$, $a,b,c\in[0,1]$ and $a+b+c=1$.

Moreover, Zhang and Fu [27] proposed a unified similarity measure for three kinds of fuzzy sets, which for the case of the intuitionistic fuzzy sets has the following form:

$$S_{ZF}\left(A,B\right)=1-\frac{1}{2n}\sum_{i=1}^{n}\left(\left|\delta_A\left(x_i\right)-\delta_B\left(x_i\right)\right|+\left|\alpha_A\left(x_i\right)-\alpha_B\left(x_i\right)\right|\right) \quad (47)$$

where

$\delta_k\left(x_i\right)=\mu_k\left(x_i\right)+\left(1-\mu_k\left(x_i\right)-v_k\left(x_i\right)\right)\mu_k\left(x_i\right)$

$\alpha_k\left(x_i\right)=v_k\left(x_i\right)+\left(1-\mu_k\left(x_i\right)-v_k\left(x_i\right)\right)v_k\left(x_i\right)$ and $k=\{A,B\}$.

Following the same procedure with their previous work [25] on Haussdorff-based similarity measures, Hung and Yang [28] suggested some similarity measures based on the distance $d_p\left(A,B\right)=\frac{1}{n}\sum_{i=1}^{n}\left(\left|\Delta_{\mu}\left(i\right)\right|^{p}+\left|\Delta_{v}\left(i\right)\right|^{p}\right)^{1/p}$ derived by using the $\boldsymbol{L}_p$ metric, defined as:

$$S_{l}^{Lp}\left(A,B\right)=\frac{2^{1/p}-d_p\left(A,B\right)}{2^{1/p}} \quad (48)$$

$$S_{e}^{Lp}\left(A,B\right)=\frac{e^{-d_p\left(A,B\right)}-e^{-\left(2^{1/p}\right)}}{1-e^{-\left(2^{1/p}\right)}} \quad (49)$$

$$S_{c}^{Lp}\left(A,B\right)=\frac{2^{1/p}-d_d\left(A,B\right)}{2^{1/p}\left(1+d_d\left(A,B\right)\right)} \quad (50)$$

with $p\ge 1$.

Furthermore, the same authors Hung and Yang [29] extended the similarity measures proposed by Pappis and Karacapilidis [52] for fuzzy sets, to IFSs and suggested two new exponential-type ones.

$$S_{w1}(A,B)=\frac{1}{n}\sum_{i=1}^{n}\frac{\min\left(\mu_A(x_i),\mu_B(x_i)\right)+\min\left(v_A(x_i),v_B(x_i)\right)}{\max\left(\mu_A(x_i),\mu_B(x_i)\right)+\max\left(v_A(x_i),v_B(x_i)\right)} \tag{51}$$

$$S_{w2}(A,B)=\frac{1}{n}\sum_{i=1}^{n}\left(1-\frac{1}{2}\left(\left|\Delta_\mu(i)\right|+\left|\Delta_v(i)\right|\right)\right) \tag{52}$$

$$S_{pk1}(A,B)=\frac{\sum_{i=1}^{n}\min\left(\mu_A(x_i),\mu_B(x_i)\right)+\min\left(v_A(x_i),v_B(x_i)\right)}{\sum_{i=1}^{n}\max\left(\mu_A(x_i),\mu_B(x_i)\right)+\max\left(v_A(x_i),v_B(x_i)\right)} \tag{53}$$

$$S_{pk2}(A,B)=1-\frac{1}{2}\left(\max_i\left(\left|\Delta_\mu(i)\right|\right)+\max_i\left(\left|\Delta_v(i)\right|\right)\right) \tag{54}$$

$$S_{pk3}(A,B)=1-\frac{\sum_{i=1}^{n}\left(\left|\Delta_\mu(i)\right|+\left|\Delta_v(i)\right|\right)}{\sum_{i=1}^{n}\left(\left|\mu_A(x_i)+\mu_B(x_i)\right|+\left|v_A(x_i)+v_B(x_i)\right|\right)} \tag{55}$$

$$S_{new1}(A,B)=1-\frac{1-\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\left(\left|\Delta_\mu(i)\right|+\left|\Delta_v(i)\right|\right)\right)}{1-\exp(-n)} \tag{56}$$

$$S_{new2}(A,B)=1-\frac{1-\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\left(\left|\sqrt{\mu_A(x_i)}-\sqrt{\mu_B(x_i)}\right|+\left|\sqrt{v_A(x_i)}-\sqrt{v_B(x_i)}\right|\right)\right)}{1-\exp(-n)} \tag{57}$$

where $\exp(x)=e^x$ is the exponential operation.

A totally different to all the previous definitions approach, was suggested by Hung and Yang [30] in order to construct novel similarity measures. They firstly defined a *divergence* measure (J_α) between two IFSs and then by using J_α they constructed a a new similarity measure.

$$S_\alpha^l(A,B)=\frac{U(\alpha)-J_\alpha(A,B)}{U(\alpha)} \tag{58}$$

$$S_\alpha^e(A,B)=\frac{e^{-J_\alpha(A,B)}-e^{-(U(\alpha))}}{1-e^{-(U(\alpha))}} \tag{59}$$

$$S_\alpha^c(A,B)=\frac{U(\alpha)-J_\alpha(A,B)}{(1+J_\alpha(A,B))U(\alpha)} \tag{60}$$

where

$$U(\alpha)=\begin{cases}\ln 2,\ \alpha=1\\ \dfrac{1}{\alpha-1}\left(1-\dfrac{1}{2^{\alpha-1}}\right), a\neq 1, \alpha>0\end{cases} \tag{61}$$

and

$$J_\alpha(A,B)=\frac{1}{n}\sum_{i=1}^{n} j_\alpha(A_i,B_i) \tag{62}$$

with

$$j_\alpha(A,B)=\begin{cases}\dfrac{-1}{\alpha-1}\left(T_{AB}^{\mu\alpha}+T_{AB}^{v\alpha}+T_{AB}^{\pi\alpha}\right), a\neq 1, \alpha>0\\ \dfrac{-1}{2}\left(L_{AB}^{\mu}+L_{AB}^{v}+L_{AB}^{\pi}\right), \qquad a=1\end{cases} \tag{63}$$

is the J_α-divergence between two intuitionistic fuzzy sets.

$T_{AB}^{q\alpha}=\left(\dfrac{q_A+q_B}{2}\right)^\alpha-\dfrac{1}{2}\left(q_A^\alpha+q_B^\alpha\right)$,

$L_{AB}^{q}=(q_A+q_B)\ln\left(\dfrac{q_A+q_B}{2}\right)-q_A\ln q_A-q_B\ln q_B$ and $q=\{\mu,\nu,\pi\}$.

Recently, Ye [31] proposed two cosine similarity measures for IFSs defined as:

$$C_{IFS}(A,B)=\frac{1}{n}\sum_{i=1}^{n}\frac{\mu_A(x_i)\mu_B(x_i)+v_A(x_i)v_B(x_i)}{\sqrt{\mu_A^2(x_i)+v_A^2(x_i)}\sqrt{\mu_B^2(x_i)+v_B^2(x_i)}} \tag{64}$$

$$W_{IFS}\left(A,B\right)=\frac{1}{n}\sum_{i=1}^{n}w_i\frac{\mu_A\left(x_i\right)\mu_B\left(x_i\right)+v_A\left(x_i\right)v_B\left(x_i\right)}{\sqrt{\mu_A^2\left(x_i\right)+v_A^2\left(x_i\right)}\sqrt{\mu_B^2\left(x_i\right)+v_B^2\left(x_i\right)}} \tag{65}$$

with $0\le w_i\le 1$ $i\in\{1,2,...,n\}$ and $\sum_{i=1}^{n}w_i=1$. If we take $w_i=1/n$, $i\in\{1,2,...,n\}$ then $W_{IFS}\left(A,B\right)=C_{IFS}\left(A,B\right)$.

Hwang and Yang [32] realized that Ye's cosine similarity measure does not satisfy condition (S1) of Definition 3 and proposed a modified version according to the following formulas:

$$S_{IFS}=\frac{1}{3}\left(C_{IFS}\left(A,B\right)+C_{IFS}^{*}\left(A,B\right)+C_{IFS}^{**}\left(A,B\right)\right) \tag{66}$$

where

$$C_{IFS}^{*}\left(A,B\right)=\frac{1}{n}\sum_{i=1}^{n}\frac{\varphi_A\left(x_i\right)\varphi_B\left(x_i\right)+v_A\left(x_i\right)v_B\left(x_i\right)}{\sqrt{\varphi_A^2\left(x_i\right)+v_A^2\left(x_i\right)}\sqrt{\varphi_B^2\left(x_i\right)+v_B^2\left(x_i\right)}} \tag{67}$$

and

$$C_{IFS}^{**}\left(A,B\right)=\frac{1}{n}\sum_{i=1}^{n}\frac{\left(1-\mu_A\left(x_i\right)\right)\left(1-\mu_B\left(x_i\right)\right)+\left(1-v_A\left(x_i\right)\right)\left(1-v_B\left(x_i\right)\right)}{\sqrt{\left(1-\mu_A\left(x_i\right)\right)^2+\left(1-v_A\left(x_i\right)\right)^2}\sqrt{\left(1-\mu_B\left(x_i\right)\right)^2+\left(1-v_B\left(x_i\right)\right)^2}} \tag{68}$$

with $\varphi_k\left(x_i\right)=\frac{1+\mu_k\left(x_i\right)-v_k\left(x_i\right)}{2}$ and $k=\{A,B\}$.

5. Discussion

It is worth noting the commonly used procedure to propose a new distance or similarity measure is to find the counter intuitive cases of the already introduced measures, followed by the development of a new one that alters this deficiency. Specific attention has to be paid to the satisfaction of the definitional axioms by the proposed measure, since these conditions determine the desirable behaviour of the measure.

The unified representation of the measurements of the previous sections, not only simplifies the overall technical presentation but also enables the direct comparison of them. For example, a closer look at d_2^p Eq.(24) and S_e^p Eq.(34) leads to the conclusion that these measures are identical, while some other measures have close relation with the Hamming distance Eq.(7). These important observations have already been reported by Baccour et al. [53] and show that some measures share the same counter intuitive cases and thus give the same unreasonable results.

Since the main point of view of this work is the performance of the distance and similarity measures in pattern recognition applications, it would be constructive to compare these measures with other representative measures of different nature. For this purpose the first information-driven measure introduced by Vlachos and Sergiadis [37] is selected. Inspired by the concept of the cross-entropy measure they proposed a symmetric discrimination information measure defined as:

$$D_{IFS}(A,B) = I_{IFS}(A,B) + I_{IFS}(B,A) \tag{69}$$

where

$$I_{IFS}(A,B) = \sum_{i=1}^{n}\left[\mu_A(x_i)\ln\frac{\mu_A(x_i)}{\frac{1}{2}\left(\mu_A(x_i)+\mu_B(x_i)\right)} + \nu_A(x_i)\ln\frac{\nu_A(x_i)}{\frac{1}{2}\left(\nu_A(x_i)+\nu_B(x_i)\right)}\right] \tag{70}$$

Besides, the comparison with the aforementioned cross-entropy measure will contribute to analyze in a more overall view the classification performance of the studied measures.

6. Performance evaluation

In order to study the ability of the proposed measures to indicate the difference or the similarity between two intuitionistic fuzzy sets, a set of experiments has been arranged. The experiments are conducted in two different directions regarding the nature of the problems where the measurements are applied.

In the most papers that introduced the measurements of the previous sections, the experimental evaluation contained was restricted in artificial classification problems. This was due to the fact that the usage of artificial intuitionistic fuzzy sets can highlight the deficiencies of the measures, although they do not have any practical meaning.

Worthwhile to point out that the reasonable behaviour of a measure, regarding the satisfaction of the definitional axioms and the avoidance of identical measurements for totally different sets, are two desirable properties. Moreover, the discrimination capability of a measure is another important property, which is very useful in pattern recognition applications. To this direction little work has been done [54,55] and more complex pattern recognition problems having big enough data have to be considered in order to exhaustively examine this property.

To this end, the analysed measures are firstly evaluated for well known from the literature artificial intuitionistic fuzzy sets, constructed to detect unreasonable behaviours. In a second experimental round the same measures are applied to recognize the samples of real pattern recognition problems. This experimental section includes examples from medical diagnosis, common benchmark pattern recognition problems and a face recognition application. The latter problem constitutes a real pattern recognition task which constitutes an important part of modern computer vision and surveillance systems and therefore the performance of the examined measures is of high importance.

In all the experiments the classification task is accomplished by a *Minimum Distance Classifier* (MDC) according to, a test sample *(TS)* is assigned to the class *(P_k)* from which its distance *(d)* is minimum and is described by the following equation:

$$k^* = \arg\min_k \left\{ d\left(P_k, TS\right) \right\} \tag{71}$$

If a similarity measure is used instead of the distance in Eq.(71), then the similarity measure has to be transformed to a distance by using the formula $d=1-S$. In this case it is expected the resulted distance measure to be normalized into [0,1]. Moreover, for the case of the problems consisting of many data samples the centre of each class, which is computed by averaging the class's feature vectors, serves as the class's prototype *(P_k)* used in Eq.(71).

In order to compare in depth the distance and similarity measures in real problems, a new performance index called *Degree of Confidence (DoC)* is introduced. This index measures the confidence of each measure to recognize a specific sample that belongs to the i^{th} class and has the following form:

$$DoC^{(i)} = \sum_{i=1, i \neq j}^{n} \left| dist\left(P_j, S\right) - dist\left(P_i, S\right) \right| \tag{72}$$

It is obvious from the above Eq.(72) that the greater $DoC^{(i)}$ the more confident the result of the specific measure is. This index is used in the next experimental sections along with the Classification Rate - CRate (Correct-Classified-Samples/Total-Samples)), in order to give a more accurate measurement of the measures' behaviour.

Worthwhile to note that the weighted measures are configured with the weights $w_i = 1/n,\ i \in \{1, 2, ..., n\}$ (so the weighted measures are equal to their non-weighted versions) and those measures that include the factor (p) are examined in two cases (p=1 & p=2) in order to investigate the dependency of the measures' performance on this free parameter. Furthermore the $\{a, b, c\}$ and $\omega_i = 1/n,\ i \in \{1, 2, ..., n\}$ parameters of the similarity measures S_C^w, S_H^w, S_L^p and S_L^p are set to {2,-1,1}, {1,1,1}, {1/3,1/3,1/3} and {1/3,1/3,1/3}, respectively.

Based on the above configuration a total of 28 distance and 45 similarity measures will participate in a large scale experimental study that aims to extract useful conclusions regarding their suitability in recognizing similar or totally different patterns.

5.1 Artificial classification problems

In a very constructive work, Li et al. [41] presented a review on similarity measures for intuitionistic fuzzy sets by investigating the particular situations where those measures give unreasonable results and finally providing the counter intuitive cases for each one of them. Although these cases are close relative to the specific measures, they constitute a very good artificial benchmark where any proposed measure should be tested. The testing IFSs used by Li et al. [41] are depicted in Table 2, while the distances' and similarities' counter intuitive cases (in bold face) for these sets are summarized in Table 3 and 4 respectively.

Table 2. Test intuitionistic fuzzy sets**.**

	Test IFSs					
	1	***2***	***3***	***4***	***5***	***6***
$A=\left[\left(x,\mu_A,\nu_A\right)\right]$	$\left[\left(x,0.3,0.3\right)\right]$	$\left[\left(x,0.3,0.4\right)\right]$	$\left[\left(x,1.0,0.0\right)\right]$	$\left[\left(x,0.5,0.5\right)\right]$	$\left[\left(x,0.4,0.2\right)\right]$	$\left[\left(x,0.4,0.2\right)\right]$
$B=\left[\left(x,\mu_B,\nu_B\right)\right]$	$\left[\left(x,0.4,0.4\right)\right]$	$\left[\left(x,0.4,0.3\right)\right]$	$\left[\left(x,0.0,0.0\right)\right]$	$\left[\left(x,0.0,0.0\right)\right]$	$\left[\left(x,0.5,0.3\right)\right]$	$\left[\left(x,0.5,0.2\right)\right]$

Table 3. Distance measures of the test sets (Table 2).

	Test IFSs							Test IFSs					
	1	***2***	***3***	***4***	***5***	***6***		***1***	***2***	***3***	***4***	***5***	***6***
d_H^1	**0.10**	**0.10**	**0.50**	**0.50**	**0.10**	**0.50**	d_E^{eh}	**0.20**	**0.10**	**1.00**	**1.00**	**0.20**	**0.10**
d_{nH}^1	**0.10**	**0.10**	**0.50**	**0.50**	**0.10**	**0.50**	d_{nE}^{eh}	**0.20**	**0.10**	**1.00**	**1.00**	**0.20**	**0.10**
d_E^1	**0.10**	**0.10**	0.71	0.50	**0.10**	0.70	d_1	**0.10**	**0.10**	**0.75**	0.50	**0.10**	**0.75**
d_{nE}^1	**0.10**	**0.10**	0.71	0.50	**0.10**	0.70	$d_2^{p=1}$	**0.10**	**0.10**	**0.50**	**0.50**	**0.10**	**0.50**
d_H^2	**0.20**	**0.10**	**1.00**	**1.00**	**0.20**	**0.10**	$d_2^{p=2}$	**0.10**	**0.10**	**0.50**	**0.50**	**0.10**	**0.50**
d_{nH}^2	**0.20**	**0.10**	**1.00**	**1.00**	**0.20**	**0.10**	d_R^{imp}	**0.06**	**0.06**	**0.00**	0.50	**0.06**	0.01
d_E^2	**0.17**	**0.10**	1.00	0.87	**0.17**	**0.10**	d_G^{imp}	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
d_{nE}^2	**0.17**	**0.10**	1.00	0.87	**0.17**	**0.10**	d_L^{imp}	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
d_H^h	**0.10**	**0.10**	1.00	0.50	**0.10**	**0.10**	d_{KD}^{imp}	**0.20**	**0.20**	**0.00**	1.00	**0.20**	0.10
d_{nH}^h	**0.10**	**0.10**	1.00	0.50	**0.10**	**0.10**	d_M^{imp}	**0.20**	**0.20**	**1.00**	**1.00**	**0.20**	0.10
d_E^h	**0.10**	**0.10**	1.00	0.50	**0.10**	**0.10**	d_{La}^{imp}	**0.14**	**0.14**	1.00	0.50	**0.14**	0.09
d_{nE}^h	**0.10**	**0.10**	1.00	0.50	**0.10**	**0.10**	d_T^{imp}	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**	**0.00**
d_H^{eh}	**0.20**	**0.10**	**1.00**	**1.00**	**0.20**	**0.10**	d^{Dimp}	**0.20**	**0.20**	**0.00**	1.00	**0.20**	0.10
d_{nH}^{eh}	**0.20**	**0.10**	**1.00**	**1.00**	**0.20**	**0.10**	D_{IFS}	**0.01**	**0.01**	**0.69**	**0.69**	**0.01**	**0.01**

Table 4. Similarity measures of the test sets (Table 2).

	Test IFSs							Test IFSs					
	1	*2*	*3*	*4*	*5*	*6*		*1*	*2*	*3*	*4*	*5*	*6*
S_C^1	**1.00**	0.90	0.50	**1.00**	**1.00**	0.95	S_{ZF}	**0.94**	0.87	**0.50**	**0.50**	**0.94**	0.95
S_C^w	**0.97**	0.90	0.67	0.83	**0.97**	**0.97**	$S_l^{Lp=1}$	**0.90**	**0.90**	**0.50**	**0.50**	**0.90**	0.95
S_H^1	**0.90**	**0.90**	**0.50**	**0.50**	**0.90**	0.95	$S_l^{Lp=2}$	**0.90**	**0.90**	0.29	**0.50**	**0.90**	0.93
S_H^w	**0.87**	**0.93**	**0.33**	**0.33**	**0.87**	**0.93**	$S_e^{Lp=1}$	**0.79**	**0.79**	**0.27**	**0.27**	**0.79**	0.89
$S_d^{p=1}$	**1.00**	0.90	0.50	**1.00**	**1.00**	0.95	$S_e^{Lp=2}$	**0.83**	**0.83**	0.17	0.33	**0.83**	0.88
$S_d^{p=2}$	**1.00**	0.90	0.50	**1.00**	**1.00**	0.95	$S_c^{Lp=1}$	**0.75**	**0.75**	**0.25**	**0.25**	**0.75**	0.86
$S_e^{p=1}$	**0.90**	**0.90**	**0.50**	**0.50**	**0.90**	0.95	$S_c^{Lp=2}$	**0.79**	**0.79**	0.15	0.29	**0.79**	0.85
$S_e^{p=2}$	**0.90**	**0.90**	0.50	**0.95**	**0.90**	**0.95**	S_{w1}	**0.75**	**0.75**	**0.00**	**0.00**	**0.75**	0.86
$S_s^{p=1}$	**0.95**	0.90	0.50	0.75	**0.95**	**0.95**	S_{w2}	**0.90**	**0.90**	**0.50**	**0.50**	**0.90**	0.95
$S_s^{p=2}$	**0.95**	0.90	0.50	0.75	**0.95**	**0.95**	S_{pk1}	**0.75**	**0.75**	**0.00**	**0.00**	**0.75**	0.86
$S_h^{p=1}$	**0.93**	**0.93**	0.50	0.67	**0.93**	0.95	S_{pk2}	**0.90**	**0.90**	**0.50**	**0.50**	**0.90**	0.95
$S_h^{p=2}$	**0.93**	**0.93**	0.50	0.67	**0.93**	0.95	S_{pk3}	**0.86**	**0.86**	**0.00**	**0.00**	**0.86**	0.92
$S_g^{p=1}$	**0.80**	**0.90**	**0.00**	**0.00**	**0.80**	**0.90**	S_{new1}	**0.85**	**0.85**	**0.38**	**0.38**	**0.85**	0.92
$S_g^{p=2}$	**0.80**	**0.90**	**0.00**	**0.00**	**0.80**	**0.90**	S_{new2}	**0.87**	**0.87**	0.38	0.20	**0.87**	0.94
$S_{\mathrm{mod},p=1}$	**0.90**	**0.90**	**0.50**	**0.50**	**0.90**	0.95	$S_{a=1}^l$	**0.97**	**0.99**	**0.00**	**0.00**	**0.97**	**0.99**
$S_{\mathrm{mod},p=2}$	**0.90**	**0.90**	**0.50**	**0.50**	**0.90**	0.95	$S_{a=2}^l$	**0.97**	**0.99**	0.00	0.25	**0.97**	**0.99**
$S_{new,p=1}$	**0.88**	**0.88**	**0.40**	**0.40**	**0.88**	0.93	$S_{a=1}^e$	**0.95**	**0.99**	**0.00**	**0.00**	**0.95**	**0.99**
$S_{new,p=2}$	**0.85**	**0.85**	**0.23**	**0.23**	**0.85**	0.93	$S_{a=2}^e$	**0.96**	**0.99**	0.00	0.21	**0.96**	**0.99**
S_l	**0.90**	**0.90**	0.00	0.50	**0.90**	**0.90**	$S_{a=1}^c$	**0.94**	0.98	**0.00**	**0.00**	**0.94**	0.99
S_e	**0.85**	**0.85**	0.00	0.38	**0.85**	**0.85**	$S_{a=2}^c$	**0.96**	**0.99**	0.00	0.18	**0.96**	**0.99**
S_c	**0.82**	**0.82**	0.00	0.33	**0.82**	**0.82**	C_{IFS}	**1.00**	0.96	**0.00**	**0.00**	0.9971	0.9965
$S_L^{p=1}$	**0.87**	**0.93**	**0.33**	**0.33**	**0.87**	**0.93**	S_{IFS}	0.9970	0.9742	**0.5690**	**0.5690**	0.9956	0.9976
$S_L^{p=2}$	**0.86**	**0.92**	0.18	0.29	**0.86**	**0.92**							

By studying Table 3, it is deduced that the most distance measures show many counter intuitive cases and therefore they failed to distinguish the IFSs accurately. However, d_E^1, d_{nE}^1 and d_{La}^{imp} distance measures have the less counter intuitive cases with the d_{La}^{imp} being the most accurate. This is justified by examining the distances for the cases 3 and 6, where d_E^1, d_{nE}^1 measures give very close values (0.71 and 0.70 for cases 3 and 6 respectively) although these cases are very different. On the other hand d_{La}^{imp} distance distinguish the two cases clearly by giving 1.00 for case 3 and 0.09 for case 6 and thus reflect the real differences of the IFSs under comparison. Furthermore, by studying carefully these two cases 3 and 6, it is deduced that the IFSs' distance of case 3 is higher than that of case 6. The aforementioned three distance measures are able to model these differences accurately, while there are distances which fail such as d_H^1, d_{nH}^1, d_1, $d_2^{p=1}$ etc.. Moreover, there are several distance measures that give identical values for some or all cases, an observation that highlights their high relevance pointed out in the discussion section.

As far as the results of the similarity measures are concerned, it is concluded that more similarity measures perform better than the distances', but there are also a lot of counter intuitive cases. The most effective similarity measure is the modified cosine similarity measure S_{IFS}, which outperforms the other measures totally. However, this measure needs to handle the $\frac{0}{0}$ indeterminacy, occurred in cases 3 and 4 in the same way with C_{IFS} due to the presence of the first factor of Eq.(66), by substituting with 0. Again, several similarity measures show an identical performance, while other measures S_C^1, $S_d^{p=1}$, $S_d^{p=2}$ (cases 1, 4, and 5) and C_{IFS} (case 1), wrongly identify different sets. Finally, it has to be noted that the free parameter *(p)* does not affect the performance of the measures, while parameter *(a)* seems to slightly influence the behaviour of the measures but without helping them to correctly measure the IFSs' differences.

5.2 Real pattern recognition problems

Although the previous experiments on artificial data highlight the behaviour of the measures under particular cases, they provide information regarding the satisfaction of the measures' fundamental definitional axioms and conditions. However, the question of how all these measures are performed in real pattern recognition applications is raised. There is not much work in this direction in the literature, and moreover there are not any comparative results between such measures in such problems.

To this end, three types of real pattern recognition applications have been selected and used to evaluate the measures under study, by pointing out some interesting outcomes.

5.2.1 Medical diagnosis

Several measures between IFSs have been applied to classify the symptoms of some patients to a set of deceases, by providing an alternative diagnosis to help doctor's decisions. A well known medical diagnosis problem, which commonly used [37,56,57] to testify the distance and/or similarity measures for IFSs is described in Table 5 & 6.

More precisely, the statement of this problem is as follows: medical diagnosis among five possible deceases of four patients (Al, Bob, Joe, Ted) by taking into account five medical symptoms (temperature, headache, stomach pain, cough, chest pain). Tables' data are presented in intuitionistic fuzzy sets and the distance and similarity measures of the previous sections are applied to make the diagnosis.

Table 5. Symptoms characteristics of each decease.

	Viral fever	Malaria	Typhoid	Stomach problem	Chest problem
Temperature	(0.4, 0.0)	(0.7, 0.0)	(0.3, 0.3)	(0.1, 0.7)	(0.1, 0.8)
Headache	(0.3, 0.5)	(0.2, 0.6)	(0.6, 0.1)	(0.2, 0.4)	(0.0, 0.8)
Stomach pain	(0.1, 0.7)	(0.0, 0.9)	(0.2, 0.7)	(0.8, 0.0)	(0.2, 0.8)
Cough	(0.4, 0.3)	(0.7, 0.0)	(0.2, 0.6)	(0.2, 0.7)	(0.2, 0.8)
Chest pain	(0.1, 0.7)	(0.1, 0.8)	(0.1, 0.9)	(0.2, 0.7)	(0.8, 0.1)

Table 6. Symptoms characteristics of each patient.

	Temperature	Headache	Stomach pain	Cough	Chest pain
Al	(0.8, 0.1)	(0.6, 0.1)	(0.2, 0.8)	(0.6, 0.1)	(0.1, 0.6)
Bob	(0.0, 0.8)	(0.4, 0.4)	(0.6, 0.1)	(0.1, 0.7)	(0.1, 0.8)
Joe	(0.8, 0.1)	(0.8, 0.1)	(0.0, 0.6)	(0.2, 0.7)	(0.0, 0.5)
Ted	(0.6, 0.1)	(0.5, 0.4)	(0.3, 0.4)	(0.7, 0.2)	(0.3, 0.4)

The diagnosis performance of each distance and similarity measure is presented in the following Table 7 and 8, respectively. In these tables the degree of confidence (DoC) for each correct diagnosis, while the symbol X indicates an incorrect diagnosis.

Table 7. Distance measures' diagnosis performance (DoC).

	Patients					Patients			
	Al	Bob	Joe	Ted		Al	Bob	Joe	Ted
d_H^1	X	5.050	3.550	2.700	d_E^{eh}	X	3.044	1.434	1.338
d_{nH}^1	X	1.010	0.710	0.540	d_{nE}^{eh}	X	1.361	0.641	0.598
d_E^1	1.511	2.807	1.734	1.393	d_1	X	1.035	0.645	0.580
d_{nE}^1	0.676	1.255	0.776	0.623	$d_2^{p=1}$	X	1.010	0.710	0.540
d_H^2	X	5.400	3.000	2.700	$d_2^{p=2}$	0.689	1.281	0.843	0.615
d_{nH}^2	X	1.080	0.600	0.540	d_R^{imp}	3.831	8.181	4.774	4.557
d_E^2	X	2.732	1.340	1.283	d_G^{imp}	5.216	**12.211**	**8.708**	**9.692**
d_{nE}^2	X	1.222	0.599	0.574	d_L^{imp}	4.047	8.528	4.727	5.981
d_H^h	X	5.300	2.900	3.100	d_{KD}^{imp}	X	8.447	4.764	3.992
d_{nH}^h	X	1.060	0.580	0.620	d_M^{imp}	4.562	8.262	4.981	4.878
d_E^h	X	3.031	1.565	1.501	d_{La}^{imp}	3.710	5.746	X	3.473
d_{nE}^h	X	1.355	0.700	0.671	d_T^{imp}	**5.594**	12.120	7.703	8.520
d_H^{eh}	X	5.400	3.000	2.700	d^{Dimp}	X	6.110	X	2.495
d_{nH}^{eh}	X	1.080	0.600	0.540	D_{IFS}	3.547	5.476	4.392	2.760

According to Eq.(72), the computation of the DoC requires the definition of the correct diagnosis for each patient. However, such important information has not been reported in the literature [37,55-57], only the performance of some similarity measures compared to the first work introduced this problem [55] has been published. However, without loss of generality, it is decided to consider the outcomes of the most recent work [37] as the correct diagnosis, which are Al (Viral fever), Bob (Stomach problem), Joe (Stomach pain) and Ted (Viral fever).

A careful study of the above results leads to the conclusion that for many distance measures is difficult to diagnose correctly Al's symptoms, while the decision of the d_T^{imp} measure is the most confident. In general the implication based distance measures outperform the measures of type I, since they provide diagnoses of high confidence, with the d_G^{imp} one being the winner in this short competition.

The corresponding results for the case of the similarity measures are similar, with many measures failing to diagnose correctly Al patient. Several measures perform satisfactory with the most efficient being $S_{new,p=2}$, followed by C_{IFS}.

Table 8. Similarity measures' diagnosis performance (DoC).

	Patients					Patients			
	Al	**Bob**	**Joe**	**Ted**		**Al**	**Bob**	**Joe**	**Ted**
S_C^1	0.700	0.990	0.690	0.620	S_{ZF}	0.744	1.199	0.927	0.652
S_C^w	0.753	1.113	0.747	0.713	$S_l^{Lp=1}$	X	1.010	0.710	0.540
S_H^1	X	1.010	0.710	0.540	$S_l^{Lp=2}$	X	0.985	0.647	0.564
S_H^w	X	0.720	0.400	0.360	$S_e^{Lp=1}$	X	1.478	0.936	0.701
$S_d^{p=1}$	0.700	0.990	0.690	0.620	$S_e^{Lp=2}$	X	1.308	0.785	0.688
$S_d^{p=2}$	0.780	1.279	0.913	0.655	$S_c^{Lp=1}$	X	1.456	0.883	0.657
$S_e^{p=1}$	X	1.010	0.710	0.540	$S_c^{Lp=2}$	X	1.338	0.768	0.674
$S_e^{p=2}$	0.689	1.281	0.843	0.615	S_{w1}	X	1.314	0.829	0.541
$S_s^{p=1}$	0.680	1.000	0.735	0.600	S_{w2}	X	1.010	0.710	0.540
$S_s^{p=2}$	0.776	1.279	0.913	0.648	S_{pk1}	X	1.533	0.978	0.616
$S_h^{p=1}$	X	0.690	0.430	0.387	S_{pk2}	0.800	2.000	X	0.850
$S_h^{p=2}$	X	0.881	0.513	0.427	S_{pk3}	X	1.259	0.890	0.575
$S_g^{p=1}$	X	1.080	0.600	0.540	S_{new1}	X	1.707	0.908	0.662
$S_g^{p=2}$	X	1.361	0.641	0.598	S_{new2}	X	1.331	0.905	0.665
$S_{\text{mod},p=1}$	X	1.010	0.710	0.540	$S_{a=1}^l$	0.465	0.862	X	0.473
$S_{\text{mod},p=2}$	0.676	1.258	0.771	0.616	$S_{a=2}^l$	X	0.744	0.504	0.392
$S_{new,p=1}$	X	2.020	1.420	1.080	$S_{a=1}^e$	0.553	1.066	X	0.592
$S_{new,p=2}$	**1.351**	**2.515**	**1.541**	**1.233**	$S_{a=2}^e$	X	0.888	0.582	0.467
S_l	X	1.060	0.580	0.620	$S_{a=1}^c$	0.593	1.182	X	0.662
S_e	X	1.272	0.634	0.685	$S_{a=2}^c$	X	0.990	0.631	0.519
S_c	X	1.312	0.621	0.674	C_{IFS}	0.962	1.433	1.130	0.729
$S_L^{p=1}$	X	0.720	0.400	0.360	S_{IFS}	0.812	1.188	0.885	0.621
$S_L^{p=2}$	X	0.997	0.489	0.469					

It is worth mentioning that the free parameters (*p* and *a)* affect significantly the performance of the measures e.g. measures $S_{\text{mod},p=1}$ and $S_{new,p=1}$ couldn't diagnose Al patient correctly but $S_{\text{mod},p=2}$ and $S_{new,p=2}$ did it. Therefore, the appropriate calibration of these free parameters may lead to more efficient measures in terms of diagnosis accuracy and decision confidence.

Comparing the best measures of each category, it can be claimed that the d_G^{imp} distance measure outperforms the similarity $S_{new,p=2}$ in terms of the degree of confidence.

5.2.3 Pattern classification benchmarks

Apart from the previous investigation of the measures' performance in a real medical diagnosis problem, two pattern classification benchmarks widely used in the literature are selected from the UCI repository [58] and used herein. The main properties of these benchmarks are summarized in the following Table 9.

Table 9. Datasets' characteristics.

Dataset	Attributes	Instances	Classes
Iris	4	150	3
Wine	13	178	3

The first dataset *Iris* consists of three different classes of Ireland flowers (*Iris Setosa*, *Iris Versicolour* and *Iris Virginica*), 50 instances for each, while 4 attributes used to describe each instance. The second dataset Wine consists of three different classes of Italian wines, variable number of instances per class (class #1: 59, class #2: 71 and class#3: 48), and 13 attributes. The measures' classification performance is summarized in the following Table 10 and 11.

Since, the original benchmarks' data are real numbers, a procedure to transform them to intuitionistic fuzzy sets need to be applied. For this purpose, the process of [37] according to, the membership and non-membership function are constructed by the following forms, is used.

$$\begin{aligned} \mu(x) &= \lambda g(x), \\ v(x) &= \left(1 - g(x)\right)^{\lambda} \end{aligned} \tag{73}$$

where $\lambda \in [0,1]$ and $g(x)$ is a fuzzy membership function. Without loss of generality, function $g(x) = -4x^2 + 4x$ is used as the fuzzy membership function and $\lambda = 1/2$. Of course, a more in depth analysis can be performed in order to find the most appropriate membership function in terms of classification accuracy, but such an analysis is beyond the scope of this work. Therefore, after data normalization into the range [0,1], the data is transformed to IFSs by using Eq.(73) with the aforementioned settings.

Table 10. Distance measures' classification performance.

	Iris Dataset		Wine Dataset			Iris Dataset		Wine Dataset	
	CRate	*DoC*	*CRate*	*DoC*		*CRate*	*DoC*	*CRate*	*DoC*
d_H^1	0.913	1.155	0.725	0.843	d_E^{eh}	0.927	0.861	0.680	0.386
d_{nH}^1	0.913	0.289	0.725	0.065	d_{nE}^{eh}	0.927	0.431	0.680	0.107
d_E^1	0.927	0.688	0.691	0.288	d_1	0.920	0.338	0.713	0.081
d_{nE}^1	0.927	0.344	0.691	0.080	$d_2^{p=1}$	0.913	0.289	0.725	0.065
d_H^2	0.920	1.543	0.708	1.262	$d_2^{p=2}$	0.900	0.320	0.702	0.072
d_{nH}^2	0.920	0.386	0.708	0.097	d_R^{imp}	**0.967**	1.707	0.674	1.256
d_E^2	0.927	0.760	0.685	0.336	d_G^{imp}	0.900	**3.248**	0.747	**3.333**
d_{nE}^2	0.927	0.380	0.685	0.093	d_L^{imp}	**0.967**	2.017	0.764	1.215
d_H^h	0.920	1.543	0.708	1.262	d_{KD}^{imp}	0.927	2.069	0.697	1.671
d_{nH}^h	0.920	0.386	0.708	0.097	d_M^{imp}	0.920	1.884	0.747	0.935
d_E^h	0.927	0.861	0.680	0.386	d_{La}^{imp}	0.893	1.502	**0.837**	0.669
d_{nE}^h	0.927	0.431	0.680	0.107	d_T^{imp}	0.947	3.199	0.629	2.204
d_H^{eh}	0.920	1.543	0.708	1.262	d^{Dimp}	0.893	1.410	0.584	0.700
d_{nH}^{eh}	0.920	0.386	0.708	0.097	D_{IFS}	0.940	0.672	0.657	0.710

From the above Table 10, it can be seen that for the Iris dataset the implication based distance measures give the highest classification rate (CRate) with the d_R^{imp} and d_L^{imp} being the most efficient measures with 96.7% classification performance. The next most efficient non implication based distance measure is D_{IFS} which classifies the Iris data with 94% accuracy (2.7% lower). For the case of the Wine data, again an implication based measure d_{La}^{imp} shows the highest classification rate of 83.7%, which is 11.2% higher than the next most efficient non implication based distance measures d_H^1 , d_{nH}^1 and $d_2^{p=1}$ (72.5%).

Table 11. Similarity measures' classification performance.

	Iris Dataset		Wine Dataset			Iris Dataset		Wine Dataset	
	CRate	*DoC*	*CRate*	*DoC*		*CRate*	*DoC*	*CRate*	*DoC*
S_C^1	0.913	0.289	**0.725**	0.065	S_{ZF}	**0.947**	0.332	0.713	0.084
S_C^w	0.920	0.322	0.719	0.075	$S_l^{Lp=1}$	0.913	0.289	**0.725**	0.065
S_H^1	0.913	0.289	0.725	0.065	$S_l^{Lp=2}$	0.920	0.308	0.713	0.072
S_H^w	0.920	0.257	0.708	0.065	$S_e^{Lp=1}$	0.913	0.516	**0.725**	0.111

$S_d^{p=1}$	0.913	0.289	**0.725**	0.065	$S_e^{Lp=2}$	0.920	0.473	0.713	0.106
$S_d^{p=2}$	0.900	0.320	0.702	0.072	$S_c^{Lp=1}$	0.913	0.557	**0.725**	0.115
$S_e^{p=1}$	0.913	0.289	**0.725**	0.065	$S_c^{Lp=2}$	0.920	0.525	0.713	0.114
$S_e^{p=2}$	0.900	0.320	0.702	0.072	S_{w1}	0.927	0.495	0.713	0.114
$S_s^{p=1}$	0.913	0.289	**0.725**	0.065	S_{w2}	0.913	0.289	**0.725**	0.065
$S_s^{p=2}$	0.900	0.320	0.702	0.072	S_{pk1}	0.920	0.531	0.719	0.121
$S_h^{p=1}$	0.920	0.225	0.713	0.054	S_{pk2}	0.900	0.485	0.624	0.139
$S_h^{p=2}$	0.927	0.251	0.685	0.060	S_{pk3}	0.920	0.353	0.719	0.087
$S_g^{p=1}$	0.920	0.386	0.708	0.097	S_{new1}	0.913	0.713	**0.725**	0.131
$S_g^{p=2}$	0.927	0.431	0.680	0.107	S_{new2}	**0.947**	**0.653**	0.685	0.133
$S_{\mathrm{mod},p=1}$	0.913	0.289	**0.725**	0.065	$S_{a=1}^{l}$	0.933	0.177	0.674	0.059
$S_{\mathrm{mod},p=2}$	0.907	0.325	0.702	0.072	$S_{a=2}^{l}$	0.927	0.136	0.685	0.045
$S_{new,p=1}$	0.913	0.577	0.725	0.130	$S_{a=1}^{e}$	0.933	0.235	0.674	0.076
$S_{new,p=2}$	0.907	0.650	0.702	**0.145**	$S_{a=2}^{e}$	0.927	0.169	0.685	0.055
S_l	0.920	0.386	0.708	0.097	$S_{a=1}^{c}$	0.933	0.275	0.674	0.088
S_e	0.920	0.513	0.708	0.123	$S_{a=2}^{c}$	0.927	0.195	0.685	0.063
S_c	0.920	0.564	0.708	0.130	C_{IFS}	0.940	0.216	0.702	0.081
$S_L^{p=1}$	0.920	0.257	0.708	0.065	S_{IFS}	0.933	0.190	0.708	0.064
$S_L^{p=2}$	0.927	0.310	0.685	0.076					

For the case of the similarity measures, S_{new2} shows the highest classification performance 94.7% with high degree of confidence (0.653) for the Iris dataset, while a lot of measures have the highest classification rate 72.5% (e.g. S_C^1, $S_d^{p=1}$, $S_e^{p=1}$) for the Wine benchmark.

What is of great importance is the superiority of the distance measures over the similarity ones in both benchmark datasets. More precisely, d_R^{imp} and d_L^{imp} distances show 2.7% higher rate than S_{new2} for the case of Iris data, while this difference is increased to 11.2% between d_{La}^{imp} and the best similarity measures for the Wine data.

5.2.4 Face recognition

Along with the previous investigation of measures' classification capabilities, an additional experiment where some human faces need to be recognized under various illumination and angle view conditions is also considered herein.

To this end distance and similarity measures are applied to recognize the faces of the well known Yale [59] face dataset consisting of 165 images of 15 persons with 11 images per person. Some samples of this dataset are illustrated in Fig.1.

Fig. 1. Samples of the Yale face dataset.

An extra pre-processing task has to be performed on the images, in order to isolate the image's part, which includes the main face's information, by discarding the useless content. For this purpose the Viola Jones face detector [60] and masking of the detected face with an ellipse, so as to remove the hair and include as much facial information is possible are applied so the images are cropped into blocks of 118×118 pixels size.

The method of orthogonal image moments is used to extract discriminative features able to distinguish the faces. Orthogonal image moments have been proved very efficient discrimination descriptors due to their ability to encode the image's content with minimum redundancy, with many applications [61-65].

Two representative moment families are applied to extract discriminative information relative to the content of the face images, namely the Zernike (ZMs) and Tchebichef (TMs) moments, which are defined in the continuous and discrete coordinate space respectively [66].

The same procedure with the previous section for the transformation of the real numbers of the discrimination features to intuitionistic fuzzy sets is also applied in the case of the moment descriptors (ZMs and TMs).

The recognition results of all distance and similarity measures for both moment features are summarized in Table 12 and 13.

Table 12. Distance measures' recognition performance.

	Zernike Moments		Tchebichef Moments			Zernike Moments		Tchebichef Moments	
	CRate	*DoC*	*CRate*	*DoC*		*CRate*	*DoC*	*CRate*	*DoC*
d_H^1	**0.800**	10.133	0.788	12.070	d_E^{eh}	0.794	4.129	0.800	5.189
d_{nH}^1	**0.800**	0.633	0.7878	0.754	d_{nE}^{eh}	0.794	1.032	0.800	1.297
d_E^1	0.782	3.448	0.806	4.243	d_1	0.788	0.691	0.794	0.844
d_{nE}^1	0.782	0.862	0.806	1.061	$d_2^{p=1}$	**0.800**	0.633	0.788	0.754
d_H^2	0.782	11.989	0.788	14.828	$d_2^{p=2}$	0.770	0.829	0.806	1.018
d_{nH}^2	0.782	0.749	0.788	0.927	d_R^{imp}	0.782	20.091	0.800	24.432
d_E^2	0.794	3.664	0.806	4.572	d_G^{imp}	0.733	**35.712**	0.806	**44.463**
d_{nE}^2	0.794	0.916	0.806	1.143	d_L^{imp}	0.770	18.426	0.782	22.224
d_H^h	0.782	11.989	0.788	14.828	d_{KD}^{imp}	0.776	25.408	**0.812**	29.786
d_{nH}^h	0.782	0.749	0.788	0.927	d_M^{imp}	0.776	21.264	0.721	25.303
d_E^h	0.794	4.129	0.800	5.189	d_{La}^{imp}	0.740	18.422	0.709	22.441
d_{nE}^h	0.794	1.032	0.800	1.297	d_T^{imp}	0.758	29.889	0.782	34.587
d_H^{eh}	0.782	11.989	0.788	14.828	d^{Dimp}	0.770	23.886	0.782	28.014
d_{nH}^{eh}	0.782	0.749	0.788	0.927	D_{IFS}	0.758	3.665	0.788	6.406

By examining the recognition results of distance measures (Table 12) it is deduced that they perform satisfactory in recognizing human faces. For the ZMs features, the classification rate varies between 73.3%-80%, with the d_H^1, d_{nH}^1 and $d_2^{p=1}$ being the most efficient and the implication based measures the most confident. On the other hand for TMs features d_{KD}^{imp} shows the highest classification accuracy of 81.2% slightly higher than that of the ZMs, with acceptable degree of confidence (29.786). Again the implication based measures proved to be high confident although in some cases the rate is lower.

Table 13. Similarity measures' recognition performance.

	Zernike Moments		Tchebichef Moments			Zernike Moments		Tchebichef Moments	
	CRate	*DoC*	*CRate*	*DoC*		*CRate*	*DoC*	*CRate*	*DoC*
S_C^1	**0.800**	0.633	0.788	0.754	S_{ZF}	0.788	0.636	0.788	0.786
S_C^w	0.794	0.672	0.794	0.816	$S_l^{Lp=1}$	0.800	0.633	0.788	0.754
S_H^1	**0.800**	0.633	0.788	0.754	$S_l^{Lp=2}$	0.794	0.648	0.788	0.778

S_H^w	0.782	0.500	0.788	0.618	$S_e^{Lp=1}$	0.800	1.273	0.788	1.423
$S_d^{p=1}$	**0.800**	0.633	0.788	0.754	$S_e^{Lp=2}$	0.794	1.093	0.788	1.253
$S_d^{p=2}$	0.770	0.829	**0.806**	1.018	$S_c^{Lp=1}$	**0.800**	1.470	0.788	1.576
$S_e^{p=1}$	**0.800**	0.633	0.788	0.754	$S_c^{Lp=2}$	0.794	1.292	0.788	1.432
$S_e^{p=2}$	0.770	0.829	**0.806**	1.018	S_{w1}	0.782	1.073	0.770	1.195
$S_s^{p=1}$	**0.800**	0.633	0.788	0.754	S_{w2}	**0.800**	0.633	0.788	0.754
$S_s^{p=2}$	0.770	0.829	**0.806**	1.018	S_{pk1}	0.794	1.158	0.776	1.298
$S_h^{p=1}$	0.788	0.461	0.794	0.562	S_{pk2}	0.745	2.032	0.721	2.402
$S_h^{p=2}$	0.788	0.621	0.800	0.770	S_{pk3}	0.794	0.668	0.776	0.796
$S_g^{p=1}$	0.782	0.749	0.788	0.927	S_{new1}	**0.800**	**3.671**	0.788	**2.882**
$S_g^{p=2}$	0.794	1.032	0.800	1.297	S_{new2}	0.788	3.401	0.776	2.724
$S_{\mathrm{mod},p=1}$	**0.800**	0.633	0.788	0.754	$S_{a=1}^l$	0.764	0.240	**0.806**	0.424
$S_{\mathrm{mod},p=2}$	0.770	0.833	**0.806**	1.026	$S_{a=2}^l$	0.794	0.188	**0.806**	0.330
$S_{new,p=1}$	**0.800**	1.267	0.788	1.509	$S_{a=1}^e$	0.764	0.328	**0.806**	0.571
$S_{new,p=2}$	0.770	1.666	**0.806**	2.052	$S_{a=2}^e$	0.794	0.236	**0.806**	0.413
S_l	0.782	0.749	0.788	0.927	$S_{a=1}^c$	0.764	0.394	**0.806**	0.678
S_e	0.782	1.089	0.788	1.297	$S_{a=2}^c$	0.794	0.276	**0.806**	0.480
S_c	0.782	1.278	0.788	1.476	C_{IFS}	0.776	0.235	0.794	0.435
$S_L^{p=1}$	0.782	0.500	0.788	0.618	S_{IFS}	0.794	0.241	0.800	0.421
$S_L^{p=2}$	0.794	0.748	**0.806**	0.933					

Concerning the performance of the similarity measures, the results are similar with that of the distances, with a lot of measures showing the highest classification rate of 80% and 80.6% for the case of ZMs and TMs features respectively. In general S_{new1} similarity measure combines the high classification rate with simultaneous high degree of confidence.

A short comparison between distance and similarities measures leads to the conclusion that the former is more appropriate to recognize human faces due to the highest or equal classification rates for both feature sets, but most of all due to the high confidence of their decisions.

7. Conclusion

An extensive review of the distance and similarity measures for intuitionistic fuzzy sets from the literature was presented in the previous sections. Through this literature review, the main characteristics of each measure were highlighted and useful conclusions regarding the relationships of these measures are drawn through a unified technical representation.

The examined measures were studied in depth, under several experimental configurations. Initially, the counter intuitive cases for each measure are extracted by using artificial intuitionistic fuzzy sets and the most robust measures that give reasonable results were pointed out.

In a second phase the aforementioned experimental analysis was focused on the performance of the measures in patter recognition applications, in order to examine their practical utility. The experiments have shown that although all the measures present a lot of counter intuitive cases, they perform well in real pattern recognition problems. More specifically, the distance measures outperform the similarity ones (the most efficient were $S_{new,p=2}$, S_{new1}, S_{new2}), with the implication based measures (e.g. d_G^{imp}) being the most accurate and confident ones, in overall.

An outcome which is of high importance is that the introduced degree of confidence (DoC) index along with the absolute classification accuracy can further help to examine the suitability of a measure.

This work aims to serve as a full guide of distance and similarity measures between IFSs for the new scientists and a basis framework for developing more robust and efficient measures for the experts in this research field.

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