A Lattice-Computing Ensemble for Reasoning Based on Formal Fusion of Disparate Data Types, and an Industrial Dispensing Application

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Abstract

By "fusion" this work means integration of disparate types of data including (intervals of) real numbers as well as possibility/probability distributions defined over the totally-ordered lattice $(\mathsf{R}, \leq)$ of real numbers. Such data may stem from different sources including (multiple/multimodal) electronic sensors and/or human judgement. The aforementioned types of data are presented here as different interpretations of a single data representation, namely Intervals' Number (IN). It is shown that the set F of INs is a partially-ordered lattice $(\mathsf{F}, \preceq)$ originating, hierarchically, from $(\mathsf{R}, \leq)$. Two sound, parametric *inclusion measure* functions $\sigma : \mathsf{F}^\mathsf{N} \times \mathsf{F}^\mathsf{N} \rightarrow [0, 1]$ result in the Cartesian product lattice $(\mathsf{F}^N, \preceq)$ towards decision-making based on reasoning. In conclusion, the space $(\mathsf{F}^N, \preceq)$ emerges as a formal framework for the development of hybrid intelligent fusion systems/schemes. A fuzzy lattice reasoning (FLR) ensemble scheme, namely *FLR pairwise ensemble*, or *FLRpe* for short, is introduced here for sound decision-making based on descriptive knowledge (rules). Advantages include the sensible employment of a sparse rule base, employment of granular input data (to cope with imprecision/uncertainty/vagueness), and employment of all-order data statistics. The advantages as well as the performance of our proposed techniques are demonstrated, comparatively, by computer simulation experiments regarding an industrial dispensing application.

Index Terms

Disparate Data Fusion, Ensemble of Experts, Fuzzy lattice reasoning (FLR), Granular data, Inclusion measure, Intervals' number (IN), Lattice-computing, Lattice theory, Sparse rules

I. Introduction

In the domain of *Soft Computing* or, equivalently, *Computational Intelligence*, the term "hybrid (system/algorithm)" frequently denotes an integration of different techniques/technologies including artificial neural networks, fuzzy systems, evolutionary/swarm computing, etc. towards improving an index of performance in real-world applications [1], [15]; the term "intelligence" is pertinent to decision-making, e.g. in pattern classification/recognition [81]; moreover, the term "(intelligent) fusion" may signify an aggregate intelligence towards improving decision-making [47]. In the aforementioned sense, a "hybrid intelligent fusion system" may be a Multiple Classifier System (MCS) [45], [48] also known in the literature as *Classifier Ensemble* [16], [58], [64], *Committee* [21], [79], or *Voting Consensus* [5], [50]. Note that a number of MCS architectures/strategies including applications have been reported [22], [28], [29], [46], [49], [51], [54], [55], [69], [70], [73], [80], [84], [85]. The MCS techniques are, typically, of statistical nature [33] in the Euclidean space R^N. Nevertheless, a "hybrid intelligent fusion system" may be defined otherwise, as explained next.

The term "fusion" may, alternatively, denote an integration of data stemming from multiple, even heterogeneous, sources including (multimodal) electronic devices as well as human judgement [6], [9], [13], [17], [20], [26], [52], [56], [63], [65], [67]. In the latter context, there is a keen interest in formal frameworks for unified decision-making based on disparate types of data that may accommodate uncertainty [9], [18], [78]. One such a framework has been proposed lately [35], in an information engineering context, based on mathematical *lattice theory* as follows.

Different authors have recognized that several types of data of practical interest, including *granules* [61], [83], are partially(lattice)-ordered [37], [71]. Hence, lattice theory emerged as a formal framework for the fusion of disparate data types [35]. In such context, *fuzzy lattice reasoning* (FLR) was originally proposed [36], [41], [43] as a specific rule-based scheme for classification in a complete lattice $(\mathsf{L}, \preceq)$ data domain including, as a special case, the lattice of hyperboxes in the Euclidean space R^N. In this work, FLR (reasoning) is defined, more widely, as any employment of an *inclusion measure* function $\sigma : \mathsf{L} \times \mathsf{L} \rightarrow [0, 1]$ for decision-making. Therefore, in the context of this work, the term "intelligent" is pertinent to "(FLR) reasoning".

Instead of a general mathematical lattice this work considers a specific one originating hierarchically from the totally-ordered lattice $(\mathsf{R}, \leq)$ of real numbers. Note that the latter (lattice) has stemmed, historically, from the conventional measurement process of successive comparisons [35], [41]. Our interest in lattice $(\mathsf{R}, \leq)$ was motivated by the existence of vast quantities of real number measurements stored worldwide. Therefore, we sought convenient data/information representations based on R. Hence, the complete lattice $(\mathsf{F}, \preceq)$ of Intervals' Numbers (IN) emerged, as detailed below, where a IN is a unified data representation including real numbers, intervals, and probability/possibility distributions [59]. In conclusion, the Cartesian product lattice $(\mathsf{F}^N, \preceq)$ is introduced here as a formal framework for developing hybrid intelligent fusion systems/schemes, where an element of lattice $(\mathsf{F}^N, \preceq)$ is interpreted here as either a rule (of a FLR scheme) or as an input to a FLR scheme.

In previous work, a FLR scheme for classification has been implemented on the σ-FLNMAP neural network architecture [35], [42], [44]. Note that the latter (neural network architecture) was introduced as an enhancement of the fuzzy-ARTMAP, or FAM for short, neural classifier [11]. More specifically, the σ-FLNMAP has extended the applicability domain of FAM from the lattice of hyperboxes in R^N to any complete lattice data domain. Moreover, even in the Euclidean space R^N, that is FAM's sole "applicability domain", classifier σ-FLNMAP has demonstrated significant improvements including tunable nonlinearities as well as the capacity to deal with both nonoverlapping hyperboxes and granular (hyperbox) input data [35], [42].

Due to the fact that both classifiers σ-FLNMAP and FAM are *unstable,* in the sense that their testing accuracy depends on the order of presenting the training data [19], [42], it turns out that both of them make good candidates for *Voting* classification schemes [10], [35], [68]. Indeed, empirical studies have clearly demonstrated an improved testing accuracy as well as a more stable testing accuracy for both FAM [3], [12], [60] and σ-FLNMAP [35], [44] in R^N. Later work has extended the applicability of σ-FLNMAP from the lattice of hyperboxes to the lattice $(\mathsf{F}, \preceq)$ of INs based on FLR [41]. In all, FLR is a *Lattice-Computing* scheme as explained next.

Lattice-Computing (LC) is a term introduced by Graña [23] to denote any computation in a mathematical lattice. Graña and colleagues have demonstrated a number of LC techniques in signal/image processing applications [24], [25]. In particular, they have employed mathematical morphology techniques in the totally-ordered lattice of real numbers. It turns out that FLR is also a LC scheme, in particular for reasoning, as shown below.

This paper is based on previously published work on FLR. The novelties of this work include the following. First, it presents a space of INs as a formal information fusion framework including a large number of references as well as pertinent discussions; a novel mathematical proof is also presented here. Second, it includes mathematical notation improvements. Third, it introduces an enhanced definition of FLR. Fourth, it demonstrates the "in principle" accommodation of granular inputs. Fifth, it introduces a novel decision-making scheme, that is a descriptive (rule-based) FLR ensemble of experts. Sixth, it shows a number of illustrative, new examples including figures. Seventh, it demonstrates preliminary (computer simulation) results regarding an industrial application.

The layout of this work is as follows. Section II presents a formal framework for fusion/integration of disparate data types. Section III describes our proposed FLR ensemble scheme. Section IV outlines an industrial application. Section V demonstrates, comparatively, preliminary experimental results. Section VI concludes by summarizing our contribution. The Appendix presents novel mathematical notation as well as a novel mathematical proof.

II. A Formal Information Fusion Framework

This section introduces constructively, in four steps, a formal information fusion framework, namely the Cartesian product lattice $(\mathsf{F}^N, \preceq)$ of Intervals' Numbers (INs). Different interpretations of INs are also presented. Note that the four-level hierarchy of lattices presented here is a novelty of this work. For the interested reader, useful notions and tools regarding lattice theory are summarized in the Appendix.

A. The Complete Lattice ($\overline{\mathsf{R}}$,$\leq$)

The set R of real numbers is a totally-ordered, non-complete lattice denoted by $(\mathsf{R}, \leq)$. It turns out that $(\mathsf{R}, \leq)$ can be extended to a complete lattice by including both symbols "$-\infty$" and "$+\infty$". In conclusion, the complete lattice $(\overline{\mathsf{R}}, \leq)$ emerges, where $\overline{\mathsf{R}} = \mathsf{R} \cup \{-\infty, +\infty\}$. Note that previous work has, erroneously, assumed that lattice $(\mathsf{R}, \leq)$ is complete [37], [59]. Even though the aforementioned error is not critical, this work considers, instead, the complete lattice $(\overline{\mathsf{R}}$,$\leq)$ [1]. We remark that complete lattices are important not only in defining an *inclusion measure* function, as shown in the Appendix, but they are also important in *mathematical morphology* [57], [66].

On the one hand, any strictly increasing function $v : \overline{\mathsf{R}} \rightarrow \mathsf{R}$ is a positive valuation in the complete lattice $(\overline{\mathsf{R}}, \leq)$. Motivated by the two constraints presented in the Appendix (subsection B), here we consider positive valuation functions $v : \overline{\mathsf{R}} \rightarrow \mathsf{R}^{\geq 0}$ such that both $v(-\infty) = 0$ and $v(+\infty) < +\infty$. On the other hand, any *bijective* (i.e. one-to-one), strictly decreasing function $\theta : \overline{\mathsf{R}} \rightarrow \overline{\mathsf{R}}$ is a dual isomorphic function in lattice $(\overline{\mathsf{R}}, \leq)$. We will refer to functions $\theta(.)$ and $v(.)$ simply as *dual isomorphic* and *positive valuation*, respectively. Useful extensions to the corresponding lattice of intervals are presented next.

B. The Complete Lattice (Δ,$\preceq$) Induced from ($\overline{\mathsf{R}}$,$\leq$)

A *generalized interval* is defined in lattice $(\overline{\mathsf{R}}, \leq)$ as follows.

Definition 1: Generalized interval is an element of the product lattice $(\overline{\mathsf{R}}, \leq^{\partial}) \times (\overline{\mathsf{R}}, \leq)$.

Recall that $\leq^{\partial}$ in Definition 1 denotes the *dual* (i.e. converse) of order relation $\leq$ in lattice $(\overline{\mathsf{R}}, \leq)$, i.e. $\leq^{\partial} \equiv \geq$. Product lattice $(\overline{\mathsf{R}}, \leq^{\partial}) \times (\overline{\mathsf{R}}, \leq) \equiv (\overline{\mathsf{R}} \times \overline{\mathsf{R}}, \geq \times \leq)$ will be denoted, simply, by $(\Delta, \preceq)$.

A generalized interval will be denoted by $[x, y]$, where $x, y \in \overline{\mathsf{R}}$. It follows that the *meet* ($\curlywedge$) and *join* ($\curlyvee$) in lattice $(\Delta, \preceq)$ are given, respectively, by $[a, b] \curlywedge [c, d] = [a \vee c, b \wedge d]$ and $[a, b] \curlyvee [c, d] = [a \wedge c, b \vee d]$.

The set of *positive* (*negative*) generalized intervals $[a, b]$, characterized by $a \leq b$ ($a > b$), is denoted by Δ_+ (Δ_-). It turns out that $(\Delta_+, \preceq)$ is a poset, namely *poset of positive generalized intervals*. Note that poset $(\Delta_+, \preceq)$ is *isomorphic* to the poset $(\tau(\overline{\mathsf{R}}), \preceq)$ of conventional intervals (sets) in $\overline{\mathsf{R}}$, i.e. $(\tau(\mathsf{R})$,$\preceq) \cong (\Delta_+$,$\preceq)$. We augmented poset $(\tau(\overline{\mathsf{R}}), \preceq)$ by a *least* (empty) interval, denoted by $O = [+\infty, -\infty]$ – We remark that a *greatest* interval $I = [-\infty, +\infty]$ already exists in $\tau(\overline{\mathsf{R}})$. Hence, the complete lattice $(\tau_O(\overline{\mathsf{R}}) = \tau(\overline{\mathsf{R}}) \cup \{O\}$,$\preceq) \cong (\Delta_+ \cup \{O\}, \preceq)$ emerged. In the sequel, we will employ isomorphic lattices $(\Delta_+ \cup \{O\}, \preceq)$ and $(\tau_O(\overline{\mathsf{R}}), \preceq)$, interchangeably. We point out that a trivial interval $[x, x]$ is an *atom* in the complete lattice $(\tau_O(\overline{\mathsf{R}}), \preceq)$, where an atom $[x, x]$ by definition satisfies both $[+\infty, -\infty] = O \prec [x, x]$ and there is no interval $[a, b] \in (\tau_O(\overline{\mathsf{R}}), \preceq)$ such that $O \prec [a, b] \prec [x, x]$.

Consider both a positive valuation function $v : \overline{\mathsf{R}} \rightarrow \mathsf{R}^{\geq 0}$ and a dual isomorphic function $\theta : \overline{\mathsf{R}} \rightarrow \overline{\mathsf{R}}$. Then, proposition 6.2 (in the Appendix) implies that function $v_{\Delta} : \Delta \rightarrow \mathsf{R}$ given by $v_{\Delta}([a, b]) = v(\theta(a)) + v(b)$ is a

[1]Personal communication with Peter Sussner in the context of the Hybrid Artificial Intelligence Systems (HAIS '2010) International Conference, 23-25 June 2010, San Sebastian, Spain. It is understood that the authors here assume full responsibility for possible errors.

positive valuation in lattice $(\Delta, \preceq)$. There follow both $v_\Delta(O = [+\infty, -\infty]) = 0$ and $v_\Delta(O = [-\infty, +\infty]) < +\infty$. Therefore, based on Theorem 6.1 (in the Appendix), the following two inclusion measures emerge in lattice $(\Delta, \preceq)$.

(1) $\sigma_\curlywedge([a, b] \preceq [c, d]) = \frac{v(\theta(a \vee c)) + v(b \wedge d)}{v(\theta(a)) + v(b)}$, and

(2) $\sigma_\curlyvee([a, b] \preceq [c, d]) = \frac{v(\theta(c)) + v(d)}{v(\theta(a \wedge c)) + v(b \vee d)}$.

The above inclusion measures are extended to the lattice $(\tau_O(\mathsf{R}), \preceq)$ of intervals (sets) as follows.

(1) $\sigma_\curlywedge([a, b] \preceq [c, d]) = \frac{v(\theta(a \vee c)) + v(b \wedge d)}{v(\theta(a)) + v(b)}$, if $a \vee c \leq b \wedge d$; otherwise, $\sigma_\curlywedge([a, b] \preceq [c, d]) = 0$, and

(2) $\sigma_\curlyvee([a, b] \preceq [c, d]) = \frac{v(\theta(c)) + v(d)}{v(\theta(a \wedge c)) + v(b \vee d)}$.

Functions $\theta(.)$ and $v(.)$ can be selected in different ways; for instance, choosing $\theta(x) = -x$ and $v(.)$ such that $v(x) = -v(-x)$ it follows $v_\Delta([a, b]) = v(b) - v(a)$. Here, we select a pair of parametric functions $v(x)$ and $\theta(x)$ so as to satisfy equality $v_\Delta([x, x]) = v(\theta(x)) + v(x) = Constant$ required for atoms by a popular FLR algorithm [42], [43]. Eligible pairs of functions $v(x)$ and $\theta(x)$ include, first, $v(x) = \frac{A}{1+e^{-\lambda(x-\mu)}}$ and $\theta(x) = 2\mu - x$, where $A, \lambda \in \mathsf{R}^{\geq 0}$, $\mu, x \in \mathsf{R}$ and, second, $v(x) = px$ and $\theta(x) = Q - qx$, where $p, q, Q > 0$, $x \in [0, A]$. Since it was assumed $v(\theta(x)) + v(x) = Constant$, for the latter pair of functions $v(x)$ and $\theta(x)$ it follows $v(\theta(x)) + v(x) = p[Q + (1 - q)x] = Constant$; therefore, $q = 1$.

C. The Complete Lattice $(\mathsf{F}, \preceq)$ Induced from $(\Delta, \preceq)$

Based on generalized interval analysis above, this subsection presents *intervals' numbers* (INs). A more general number type is defined in the first place, next.

Definition 2: Generalized interval number, or *GIN* for short, is a function $G : (0, 1] \to \Delta$.

Let G denote the set of GINs. It follows complete lattice $(\mathsf{G}, \preceq)$, as the Cartesian product of complete lattices $(\Delta, \preceq)$. Our interest here focuses on the *sublattice*[2] of *intervals' numbers* defined next.

Definition 3: An *Intervals' Number*, or *IN* for short, is a GIN F such that both $F(h) \in (\Delta_+ \cup \{O\})$ and $h_1 \leq h_2 \Rightarrow F(h_1) \succeq F(h_2)$.

Let F denote the set of INs. It follows that $(\mathsf{F}, \preceq)$ is a complete lattice with least element $O = O(h) = [+\infty, -\infty]$, $h \in (0, 1]$ and greatest element $I = I(h) = [-\infty, +\infty]$, $h \in (0, 1]$. Conventionally, a IN will be denoted by a capital letter in italics, e.g. $F \in \mathsf{F}$.

Definition 3 implies that a IN F is a function from interval $(0, 1]$ to the set $\tau(\overline{\mathsf{R}}) \cup \{[+\infty, -\infty]\}$ of intervals, i.e. $F(h) = [a_h, b_h]$, $h \in (0, 1]$, where both interval-ends a_h and b_h are functions of $h \in (0, 1]$.

The following two inclusion measures emerge, respectively, in the complete lattice $(\mathsf{F}, \preceq)$ of INs [34], [35]:

(1) $\sigma_\curlywedge(F_1 \preceq F_2) = \int_0^1 \sigma_\curlywedge(F_1(h) \preceq F_2(h))dh$.

(2) $\sigma_\curlyvee(F_1 \preceq F_2) = \int_0^1 \sigma_\curlyvee(F_1(h) \preceq F_2(h))dh$.

[2] A *sublattice* of a lattice $(\mathsf{L}, \preceq)$ is another lattice $(\mathsf{S}, \preceq)$ such that $\mathsf{S} \subseteq \mathsf{L}$.

The following Proposition derives from [37].

Proposition 2.1: Consider a continuous *dual isomorphic* function $\theta : \overline{\mathsf{R}} \to \overline{\mathsf{R}}$ and a continuous *positive valuation* function $v : \overline{\mathsf{R}} \to \mathsf{R}^{\geq 0}$. Let $X_0(h) = [x_0, x_0]$, $h \in (0,1]$ be a trivial (point) IN, moreover let $E(h)$, $h \in (0,1]$ be a IN with *upper-semicontinuous* membership function $m_E : \mathsf{R} \to \mathsf{R}$. Then $\sigma_{\curlywedge}(X_0 \preceq E) = m_E(x_0)$.

We remark that Proposition 2.1 couples a IN's two different representations, namely the *interval-representation* and the *membership-function-representation*. The principal advantage of the former (interval) representation is that it enables useful algebraic operations, whereas the principal advantage of the latter (membership function) representation is that it enables convenient interpretions, e.g. fuzzy logic interpretions, etc.

D. Extensions to More Dimensions

A N-tuple IN will be denoted by a capital letter in bold, e.g. $\mathbf{F} = (F_1, ..., F_N) \in \mathsf{F}^N$. Lattice $(\mathsf{F}^N, \preceq)$ is the "fourth level" in a hierarchy of complete lattices whose "first level", "second level" and "third level" include lattices ($\overline{\mathsf{R}}$,$\leq$), (Δ,$\preceq$) and $(\mathsf{F}, \preceq)$, respectively.

The following Proposition derives from [37].

Proposition 2.2: Consider N complete lattices $(\mathsf{L}_i, \preceq)$, $i \in \{1, ..., N\}$ each one equipped with an inclusion measure function $\sigma_i : \mathsf{L}_i \times \mathsf{L}_i \to [0,1]$, respectively. Consider N-tuples $\mathbf{x} = (x_1, \ldots, x_N)$ and $\mathbf{y} = (y_1, \ldots, y_N)$ in $\mathsf{L} = \mathsf{L}_1 \times \cdots \times \mathsf{L}_N$. Furthermore, consider the conventional lattice ordering $\mathbf{x} \preceq \mathbf{y} \Leftrightarrow x_i \preceq y_i, \forall i \in \{1, ..., N\}$. Then, both functions (1) $\sigma_\wedge : \mathsf{L} \times \mathsf{L} \to [0,1]$ given by $\sigma_\wedge(\mathbf{x} \preceq \mathbf{y}) = \min\limits_{i \in \{1,\ldots,N\}} \{\sigma_i(x_i \preceq y_i)\}$, and (2) $\sigma_\Pi : \mathsf{L} \times \mathsf{L} \to [0,1]$ given by $\sigma_\Pi(\mathbf{x} \preceq \mathbf{y}) = \prod\limits_{i \in \{1,\ldots,N\}} \sigma_i(x_i \preceq y_i)$, are inclusion measures in lattice $(\mathsf{L}, \preceq)$.

We remark that Propositions 2.1 and 2.2 establish that, for trivial inputs, an inclusion measure reduces to standard fuzzy inference system (FIS) practices [37].

E. IN Interpretations, Representation Issues & More, Useful Results

The complete lattice $(\mathsf{F}, \preceq)$ of INs has been studied in a series of publications [34], [38], [40], [41], [59], [62]. In particular, it has been shown that a IN is a mathematical object, which may admit different interpretations as follows. First, based on the "resolution identity theorem" [82], a IN $F(h)$, $h \in (0,1]$ may be interpreted as a fuzzy number, where $F(h)$ is the corresponding α-cut for $\alpha = h$. Hence, a IN $F : (0,1] \to \tau_O(\mathsf{R})$ may, equivalently, be represented by an *upper-semicontinuous* membership function $m_F : \mathsf{R} \to (0,1]$ – Note that a number of authors have employed α-cuts and/or intervals in fuzzy logic applications [2], [74], [75], [76], [77]. There follows equivalence $m_{F_1}(x) \leq m_{F_2}(x) \Leftrightarrow F_1(h) \preceq F_2(h)$, where $x \in \mathsf{R}$, $h \in (0,1]$ [59]. Second, a IN $F(h)$, $h \in (0,1]$ may also be interpreted as a probability distribution such that interval $F(h)$ includes $100(1-h)\%$ of the distribution, whereas the remaining $100h\%$ is split even both below and above interval $F(h)$.

Fig.1 explains how a IN can be constructed from a population of (real number) data samples using algorithm CALCIN [34], [35], [39], [59], [62]. More specifically, Fig.1(a) displays the data itself. Fig.1(b) displays a histogram of the data in Fig.1(a) in 10 steps of length $\Delta x = 0.04$. Hence, the histogram of Fig.1(b) may be thought of as a *probability density function* (pdf) approximation, which (histogram) asymptotically tends to the corresponding pdf when both $\Delta x \to 0$ and the number of data samples tends to infinity. Fig.1(c) displays the corresponding cumulative distribution function (PDF). Finally, Fig.1(d) displays a IN computed from the PDF of Fig.1(c) using the algebraic formulas shown within Fig.1(d); that is, algorithm CALCIN.

Fig.2 shows the two different representations of the IN (F) computed in Fig.1(d). More specifically, Fig.2(a) displays the membership-function-representation of IN F, whereas Fig.2(b) displays the corresponding interval-representation for $L = 32$ different levels spaced evenly over the interval $(0, 1]$. Triangular INs are of particular significance in practice, therefore they are studied next.

Consider both the triangular IN F, with membership function $m_F(x)$, and the trivial IN V_0 in Fig.3. IN F is specified by the three parameters m, w_L and w_R. A horizontal line at height $h \in (0, 1]$ intersects IN F at points a_h and b_h; moreover, it intersects trivial IN V_0 at points c_h and d_h, where $c_h = d_h = V_0$. Since the left line of the triangular membership function $m_F(x)$ equals $y = [x - (m - w_L)]/w_L$ and the right line of $m_F(x)$ equals $y = [(m + w_R) - x]/w_R$, it follows $a_h = w_L h + (m - w_L)$, moreover $b_h = -w_R h + (m + w_R)$. Next, we analytically calculate inclusion measure *sigma-join* $\sigma_{\curlyvee}(F \preceq V_0) = \int\limits_0^1 \frac{v(\theta(c_h))+v(d_h)}{v(\theta(a_h \wedge c_h))+v(b_h \vee d_h)} dh$ using $v(x) = px$ and $\theta(x) = Q - x$. Integral $\int \frac{1}{ax+b} dx = \frac{1}{a} ln|ax + b| + C_0$ will be useful in the following calculations.

(1) For $m + w_R \leq V_0$, it follows

$$\sigma_{\curlyvee}(F \preceq V_0) = \int\limits_0^1 \frac{Q-c_h+d_h}{Q-a_h+d_h} dh = -Q \int\limits_0^1 \frac{1}{w_L h+[(m-w_L)-(Q+V_0)]} dh = \frac{Q}{w_L} ln \frac{(Q+V_0)-m+w_L}{(Q+V_0)-m}.$$

(2) For $m \leq V_0 \leq m + w_R$, it follows

$$\sigma_{\curlyvee}(F \preceq V_0) = \int\limits_0^{h_0} \frac{Q-c_h+d_h}{Q-a_h+b_h} dh + \int\limits_{h_0}^1 \frac{Q-c_h+d_h}{Q-a_h+d_h} dh = -Q \int\limits_0^{h_0} \frac{1}{(w_L+w_R)h-(Q+w_L+w_R)} dh - Q \int\limits_{h_0}^1 \frac{1}{w_L h-[Q-(m-w_L)+V_0]} dh =$$
$$\frac{Q}{w_L+w_R} ln \frac{Q+w_L+w_R}{(Q+w_L+w_R)-(w_L+w_R)h_0} + \frac{Q}{w_L} ln \frac{[Q-(m-w_L)+V_0]-w_L h_0}{[Q-(m-w_L)+V_0]-w_L},$$ where $h_0 = m_F(V_0)$.

(3) For $m - w_L \leq V_0 \leq m$, it follows

$$\sigma_{\curlyvee}(F \preceq V_0) = \int\limits_0^{h_0} \frac{Q-c_h+d_h}{Q-a_h+b_h} dh + \int\limits_{h_0}^1 \frac{Q-c_h+d_h}{Q-c_h+b_h} dh = -Q \int\limits_0^{h_0} \frac{1}{(w_L+w_R)h-(Q+w_L+w_R)} dh - Q \int\limits_{h_0}^1 \frac{1}{w_R h-[Q-V_0+(m+w_R)]} dh =$$
$$\frac{Q}{w_L+w_R} ln \frac{Q+w_L+w_R}{(Q+w_L+w_R)-(w_L+w_R)h_0} + \frac{Q}{w_R} ln \frac{[Q-V_0+(m+w_R)]-w_R h_0}{[Q-V_0+(m+w_R)]-w_R},$$ where $h_0 = m_F(V_0)$.

(4) For $V_0 \leq m - w_L$, it follows

$$\sigma_{\curlyvee}(F \preceq V_0) = \int\limits_0^1 \frac{Q-c_h+d_h}{Q-c_h+b_h} dh = -Q \int\limits_0^1 \frac{1}{w_R h-[Q-V_0+(m+w_R)]} dh = \frac{Q}{w_R} ln \frac{(m+Q-V_0)+w_R}{m+Q-V_0}.$$

A triangular IN's edge corresponds to a uniform pdf as shown in Fig.4(a) as well as in Fig.4(b). Let $p_1(x)$ and $p_2(x)$ be the latter pdfs corresponding to INs F_1 and F_2, respectively. More specifically, it is

$$p_i(x) = \begin{cases} \frac{1}{2w_L}, & m_i - w_L \leq x \leq m_i \\ \frac{1}{2w_R}, & m_i \leq x \leq m_i + w_R \end{cases}$$, for $i \in \{1, 2\}$,

where w_L and w_R represent the ranges of the uniform pdf located to the left and to the right, respectively, of the median m_i, $i \in \{1, 2\}$; hence, in Fig.4(a) it is $w_L = r$, $w_R = R$, whereas in Fig.4(b) it is $w_L = R$, $w_R = r$. Note that the *median* "m" of a pdf $p(x)$ is defined here such that $\int\limits_{-\infty}^{m} p(x)dx = 0.5 = \int\limits_{m}^{+\infty} p(x)dx$. Next, we compute the means as well as the variances of pdfs $p_1(x)$ and $p_2(x)$ corresponding to the INs F_1 and F_2, respectively.

$$\mu_1 = \int\limits_{-\infty}^{+\infty} x p_1(x) dx = \int\limits_{m_1-r}^{m_1} x \frac{1}{2r} dx + \int\limits_{m_1}^{m_1+R} x \frac{1}{2R} dx = m_1 + \frac{R-r}{4}.$$

$$\mu_2 = \int_{-\infty}^{+\infty} x p_2(x)dx = \int_{m_2-R}^{m_2} x \frac{1}{2R} dx + \int_{m_2}^{m_2+r} x \frac{1}{2r} dx = m_2 - \frac{R-r}{4}.$$

$$\sigma_1^2 = \int_{-\infty}^{+\infty} (x-\mu_1)^2 p_1(x)dx = \int_{m_1-r}^{m_1} (x-\mu_1)^2 \frac{1}{2r} dx + \int_{m_1}^{m_1+R} (x-\mu_1)^2 \frac{1}{2R} dx = \frac{5r^2+5R^2+6Rr}{48}.$$

$$\sigma_2^2 = \int_{-\infty}^{+\infty} (x-\mu_2)^2 p_2(x)dx = \int_{m_2-R}^{m_2} (x-\mu_2)^2 \frac{1}{2R} dx + \int_{m_2}^{m_2+r} (x-\mu_2)^2 \frac{1}{2r} dx = \frac{5r^2+5R^2+6Rr}{48}.$$

We remark that $w_L = w_R$ implies both $\mu = \int_{-\infty}^{+\infty} x p(x)dx = \int_{m-w_L}^{m+w_R} x \frac{1}{w_L+w_R} dx = m$ and $\sigma^2 = \frac{(w_L+w_R)^2}{12}$ as expected for a uniform pdf – Recall also that a uniform pdf corresponds to an isosceles triangular IN [34], [35].

In Fig.4(c), pdfs $p_1(x)$ and $p_2(x)$ were placed such that $\mu_1 = \mu = \mu_2$; the corresponding INs, respectively, F_1 and F_2 are also shown in Fig.4(c). On the one hand, note that both the first- and the second- order statistics of pdfs $p_1(x)$ and $p_2(x)$ are identical, i.e. $\mu_1 = \mu_2$ and $\sigma_1 = \sigma_2$. Nevertheless, pdfs $p_1(x)$ and $p_2(x)$ differ in their third-order statistic, namely their *skewness*. More specifically, $p_1(x)$ is skewed to the left, whereas $p_2(x)$ is skewed to the right. On the other hand, recall that an inclusion measure function can detect all-order statistics [39], [40], [41]. Hence, in Fig.4(c), an inclusion measure can discriminate between INs F_1 and IN F_2 induced from pdfs $p_1(x)$ and $p_2(x)$, respectively, as demonstrated below.

Furthermore, let us define the following two alternative conditions/specifications (S1) $|m_i - V_0| \leq T$ and (S2) $|\mu_i - V_0| \leq T$, for a user-defined threshold value T, where V_0 and m_i, μ_i for $i \in \{1,2\}$ as well as R, r are shown in Fig.4. From both Fig.4(a) and Fig.4(b) it follows that exactly 0.5 of the distribution does not satisfy (S1). Moreover, first, from Fig.4(a) it follows that $0.5+(R-r)/8R$ of the distribution does not satisfy (S2) and, second, from Fig.4(b) it follows that $0.5-(R-r)/8R$ of the distribution does not satisfy (S2). Note also that the truth of inequality $m_i < \mu_i$ ($m_i > \mu_i$) indicates that the corresponding pdf is skewed to the left (right).

III. A Fuzzy Lattice Reasoning (FLR) Ensemble Scheme

Fuzzy lattice reasoning (FLR) is a term proposed originally for a concrete classification scheme [43], where an inclusion measure function $\sigma(A \preceq B)$ was employed, in the lattice of hyperboxes in R^N, to compute a (fuzzy) degree of inclusion of a hyperbox A to another one B. It was also shown that an inclusion measure $\sigma(.,.)$ supports two different modes of reasoning, namely *Generalized Modus Ponens* and *Reasoning by Analogy*. More specifically, on the one hand, *Generalized Modus Ponens* is supported as follows: Given both a rule "IF *variable* V_0 is E THEN *proposition* p" and a proposition "*variable* V_0 is E_p" such that $E_p \preceq E$, where both E_p and E are elements in a lattice $(\mathsf{L}, \preceq)$, it reasonably follows "*proposition* p". On the other hand, *Reasoning by Analogy* is supported as follows: Given both a set of rules "IF *variable* V_0 is E_k THEN *proposition* p_k", $k \in \{1,\ldots,K\}$ and a proposition "*variable* V_0 is E_p" such that $E_p \npreceq E_k$, for $k \in \{1,\ldots,K\}$, it follows "*proposition* p_J", where $J \doteq arg \max_{k\in\{1,\ldots,K\}} \{\sigma(E_p \preceq E_k) < 1\}$.

A FLR extension to the lattice of INs has been possible according to the following rationale. We know (see in section II-C) that a IN can, equivalently, be represented either by a membership function or by a set of intervals. Therefore, since an interval is a hyperbox in space R^1, it follows that an inclusion measure function can be extended from space R^1 to the space F of INs by a single integral operation. Further enhancements are proposed next.

A. FLR Enhancements

Here we propose using the term FLR to denote any decision-making based on an inclusion measure function $\sigma(.,.)$. Note that advantages of using an inclusion measure $\sigma(.,.)$ include, first, accommodation of nontrivial (granular) input data, second, activation of a rule by an input outside the rule's support (hence, a *sparse* rule-base becomes "sensibly" usable) and, third, a capacity to employ alternative positive valuation functions than $v(x) = x$ (the latter positive valuation is exclusively employed in the literature, implicitly). We point out that a *parametric* positive valuation function may introduce tunable nonlinearities by optimal parameter estimation techniques; likewise, for a *parametric* dual isomorphic function.

Recent work [37] has demonstrated that conventional fuzzy inference systems (FISs) [27], [30], [53], [72] apply "in principle" FLR, in lattice $(\mathsf{F}^N, \preceq)$, as follows.

A FIS, typically, includes K rules $R_k, k = 1, \ldots K$, of the following form

Rule R_k : IF (*variable* V_1 is $F_{k,1}$).AND. . . . .AND.(*variable* V_N is $F_{k,N}$) THEN *proposition* p_k,

where the antecedent of rule R_k is the conjunction of N simple propositions "*variable* V_i is $F_{k,i}$", $i = 1, \ldots N$, moreover the consequent "*proposition* p_k" of rule R_k is typically either a likewise proposition (e.g. in a Mamdani type FIS [53]) or a polynomial (e.g. in a Sugeno type FIS [72]). Our interest here focuses on rule antecedents. In particular, we assume that the degree of activation of a simple proposition "*variable* V_i is $F_{k,i}$", $i = 1, \ldots N$ by another one "*variable* V_i is $F_{0,i}$" equals $\sigma_{\curlyvee}(F_{0,i} \preceq F_{k,i})$. The following examples demonstrate some technical application details.

B. FLR Examples in lattice $(\mathsf{F}, \preceq)$

In this work we employ solely inclusion measure $\sigma_{\curlyvee}(.,.)$ rather than $\sigma_{\curlywedge}(.,.)$ because only inclusion measure $\sigma_{\curlyvee}(.,.)$ is non-zero for non-overlapping INs; hence, only $\sigma_{\curlyvee}(.,.)$ can reason based on a *sparse* rule base.

Example - 1

INs F and V_0 referred to, in this example, are shown in Fig.3.

Fig.5 plots inclusion measure $\sigma_{\curlyvee}(F \preceq V_0)$ versus the median m of IN F from $m = 0.5$ to $m = 9.5$ using parameter values $w_L = w_R = 0.5$ and $V_0 = 4.6$; moreover, both the linear positive valuation $v(x) = px$ and dual isomorphic function $\theta(x) = Q - x$ were used with parameter values $p = 1$, $Q = 10$. Equality $w_L = w_R = 0.5$ implies that triangular IN F has, in particular, an isosceles triangular shape – Recall that an isosceles triangular IN corresponds to a uniform pdf. Since the median (m) equals the mean (μ) of a uniform pdf it follows that, for an isosceles triangular IN, the x-axis in both Fig.5 and Fig.6, denotes m as well as μ.

Fig.6 plots inclusion measure $\sigma_{\curlyvee}(F \preceq V_0)$ versus its median m from $m = 0.5$ to $m = 9.5$ using parameter values $w_L = w_R = 0.5$ and $V_0 = 4.6$. Moreover, both positive valuation $v(x) = \frac{1}{1+e^{-0.5(x-4.6)}}$ and dual isomorphic function $\theta(x) = 2(4.6) - x$ were employed.

Notice the similarity of Fig.5 and Fig.6, where each figure was generated using a different positive valuation function $v(x)$. In particular, Fig.5 was generated using a *linear* positive valuation, whereas Fig.6 was generated using a *sigmoid* positive valuation. In all our experiments, in the context of this work, we empirically confirmed that for any linear positive valuation $v_\ell(x)$ there is a sigmoid positive valuation $v_s(x)$, which produces an “identical”, for all practical purposes, inclusion measure $\sigma_\curlyvee(.,.)$ function. A sigmoid positive valuation is preferable because it is defined over the whole set R of real numbers, therefore no truncation/normalization is necessary. In conclusion, unless otherwise specified, in the remaining of this work we employ sigmoid positive valuation functions.

Example - 2

The previous example has dealt with isosceles (triangular) INs. This example considers non-isosceles triangular IN shapes towards demonstrating that an inclusion measure can effectively detect higher-order statistics.

Fig.7(a) displays inclusion measure $\sigma_\curlyvee(F_1 \preceq V_0)$ versus its median m_1 from $m_1 = 3$ to $m_1 = 90$ using IN F_1 parameter values $w_L = r = 3$, $w_R = R = 10$ and $V_0 = 65$; Fig.7(b) shows the latter figure in the vicinity of its global maximum at $m_1 = 65$. Likewise, Fig.7(c) displays inclusion measure $\sigma_\curlyvee(F_2 \preceq V_0)$ versus its median m_2 from $m_2 = 10$ to $m_2 = 97$ using IN F_2 parameter values $w_L = R = 10$, $w_R = r = 3$ and $V_0 = 65$; Fig.7(d) shows the latter figure in the vicinity of its global maximum at $m_2 = 65$. Where, INs F_1, F_2 and V_0 are shown in Fig.4. Finally, Fig.7(e) displays both inclusion measures $\sigma_\curlyvee(F_1 \preceq V_0)$ and $\sigma_\curlyvee(F_2 \preceq V_0)$ versus their (identical) mean μ. More specifically, Fig.7(e) demonstrates that $\sigma_\curlyvee(F_2 \preceq V_0)$ reaches its global maximum *before* $V_0 = 65$, as expected, because IN F_2 is skewed to the right; whereas, $\sigma_\curlyvee(F_1 \preceq V_0)$ reaches its global maximum *after* $V_0 = 65$, also as expected, because IN F_1 is skewed to the left.

C. *FLRpe: A Pairwise FLR Ensemble Scheme for Reasoning*

Based on an expert-supplied proposition p : “Variable V equals x” the question here is to decide whether another proposition p_0 : “Variable V equals x_0” is true or not, where both x and x_0 are INs. We responded to the aforementioned question by computing a (fuzzy) degree of fulfillment of implication “$p \rightarrow p_0$” by $\sigma_\curlyvee(x \preceq x_0)$. More specifically, if $\sigma_\curlyvee(x \preceq x_0) \geq T$, where $T \in [0, 1]$ is user-defined, only then proposition p_0 is accepted.

Since a single expert proposition p may be prone to errors, hence it may be unreliable, we assumed an ensemble of N experts each one of whom supplied one proposition p_k : “Variable V equals x_k’, $k \in \{1, \ldots, N\}$. Our basic assumption is that at least 2 out of the N experts are reliable. In conclusion, FLR is carried out by considering all different pairs of experts as shown in Algorithm 1, that is the FLRpe scheme.

We remark that the FLRpe scheme accepts proposition p_0 if and only if the corresponding implications $p_k \rightarrow p_0$, $k \in \{1, \ldots N\}$ of any two experts $k \in \{i, j\}$ are jointly accepted, in the sense that it is $\sigma_\curlyvee(x_k \preceq x_0) \geq T$ for two different experts $k \in \{i, j\}$ as indicated in the mathematical expression in the last step of Algorithm 1; the latter (expression) derives from Proposition 2.2. In other words, proposition p_0 is accepted if and only if the maximum ($\bigvee$) inclusion measure $\sigma_\curlyvee(.)$ of all different pairs of experts is above a user-defined threshold $T \in [0, 1]$. Apparently, the FLRpe is a “collective reasoning” scheme based on an ensemble of experts.

Algorithm 1 FLRpe: A Pairwise FLR Ensemble Scheme

1: Consider a proposition p_0 : "Variable V equals x_0" and a threshold $T \in [0,1]$. Furthermore, consider N expert-supplied propositions p_k : "Variable V equals x_k", $k \in \{1,\ldots,N\}$, where x_0, x_k are INs, $k \in \{1,\ldots,N\}$.
2: Consider one implication r_k, $k \in \{1,\ldots,N\}$ per expert as follows:
Implication r_k : IF p_k THEN p_0, symbolically $p_k \rightarrow p_0$.
3: Compute the degree $\sigma_\curlyvee(x_k \preceq x_0)$ of fulfillment of each implication $r_k : p_k \rightarrow p_0$, $k \in \{1,\ldots,N\}$.
4: Accept proposition p_0 if and only if
$$\bigvee_{i,j\in\{1,\ldots,N\},i\neq j} \sigma_\wedge([x_i,x_j] \preceq [x_0,x_0]) = \bigvee \{ \bigwedge_{i,j\in\{1,\ldots,N\},i\neq j} \{\sigma_\curlyvee(x_i \preceq x_0), \sigma_\curlyvee(x_j \preceq x_0)\} \} \geq T$$

IV. An Industrial Dispensing Application

This section outlines an industrial application.

A. The Industrial Problem

Ouzo is a popular Greek liquor, whose final stage production involves dispensing three different liquids, namely *water*, *spirit*, and *yeast*, to a "mixing" tank. More specifically, *water* is typically supplied by a local utility company, *spirit* is a commercial product whose $G^s = 96\%$ volume is pure ethanol, moreover the *yeast*, whose G^y volume (in the range $40\% - 80\%$) is pure ethanol, is prepared according to a local recipe.

The Greek law calls for a specific percentage (G_1^b) of ethanol in the final (ouzo) product, e.g. $G_1^b = 38\%$ or $G_1^b = 40\%$, etc. Furthermore, the law calls for a specific ratio $p_1^y : p_1^s$, where p_1^y denotes the final product's ethanol percentage stemming-from-yeast and p_1^s denotes the corresponding percentage stemming-from-commercial-spirit; it is $p_1^y + p_1^s = 1$. In the context of this work, we call pair $(G_1^b, p_1^y : p_1^s)$ *alcoholic identity* of the (ouzo) product. Currently, the production of ouzo is largely empirical, therefore it is prone to errors as explained next.

Typically, a skilled worker (manually) calculates the volumes of water (V_1^w), spirit (V_1^s), and yeast (V_1^y) required to produce a specific volume V_1^b of ouzo of *alcoholic identity* $(G_1^b, p_1^y : p_1^s)$. Nevertheless, when a different volume $V_2^b \neq V_1^b$ is requested, at the absence of a skilled worker to compute the corresponding volumes V_2^w, V_2^s, and V_2^y, then errors may occur. Another source of errors regards the manual dispensing of volumes V_1^w, V_1^s, and V_1^y to the mixing tank. Hence, the *alcoholic identity* of the final (ouzo) product might be outside specifications. It is of practical interest to keep, an automated ouzo production, within specifications.

Work is, currently, under way towards automating the production of ouzo for a local beverage company in the Greek Macedonia region. Note that the problem of industrial dispensing has been treated also by other authors [14] using conventional modeling techniques; moreover, fuzzy regression techniques have been employed [32]. We applied the FLRpe scheme via a novel software platform, developed for the needs of this work as described next.

B. A Novel Software Platform

A novel software platform, namely XtraSP.v1 (Fig.8), was developed for the needs of this work using the Labview environment of the National Semiconductors Company. XtraSP.v1 operates as a user-friendly interface for controlling all the required electromechanical equipment, including four valves and one pump, via a NI USB-6501 device. The latter (USB) is a Universal Serial Bus to digital I/O device which also measures the flow, in the range $6 - 120\ \ell t/min$, to the mixing tank by counting pulses generated by a *flowmeter* using a 32 bit long counter. Mounted (inside) on the upper side of the mixing tank there is an *ultrasonic level meter* (U.L.M.) device, which measures the liquid level in the mixing tank with accuracy in the range $3 - 6\ mm$ by transmitting short ultrasonic pulses to the liquid surface. In addition, there is a transparent *communicating tube* (C.T.) connected to the side of the mixing tank, which (tube) functions as an indicator of the liquid level (in the mixing tank) by operating on the principle of communicating tubes. The overall physical system architecture is shown in the upper half of Fig.8.

In worksheet cells of XtraSP.v1 a user can specify (a) A label, e.g. for a tank, (b) An initial quantity of a liquid in a tank, (c) The percentage of ethanol in both the (commercial) spirit and the yeast, (d) The total percentage of ethanol in the undisposed ouzo, (e) The percentages of ethanol in the undisposed ouzo stemming, respectively, from (commercial) spirit and yeast, (f) The desired percentage of (pure) ethanol in the mixing tank, (g) The desired percentages of ethanol in the mixing tank stemming, respectively, from (commercial) spirit and yeast. Box "DECISION-SUPPORT & PARAMETERS" allows the user to specify useful rules & parameters.

Software platform XtraSP.v1 can automatically carry out any required calculation/action on user demand. Furthermore, a number of safety instructions as well as warning messages can be issued. Note also that software platform XtraSP.v1 can operate either in a SIMULATION mode or in a real-world OPERATION mode, where the latter (mode) can be either MANUAL or AUTOMATIC.

C. Implementation of the FLRpe Scheme

An expert-based reasoning scheme, which may also accommodate uncertainty/ambiguity, is of particular interest in an industrial application. Furthermore, the capacity to effectively cope with an unreliable expert is a specification of critical importance because an unreliable expert may result in a final product outside specifications. The proposed FLRpe scheme appears to satisfy the aforementioned specifications, therefore it was applied as described next.

The volume of a liquid being dispensed to the mixing tank was estimated simultaneously by three different "experts" including, first, a flowmeter measurement device, second, an ultrasonic level meter measurement device and, third, a human expert who visually consults the transparent tube connected to the side of the mixing tank. We employed the following (binary) decision rule.

Rule R : IF volume v (of the liquid being dispensed) equals V_0 THEN *stop dispensing*,

We assumed that the degree of truth of a Rule R equals the degree of truth of its antecedent. Hence, we "*stop dispensing*" if the antecedent proposition p_0: "volume v (of the liquid being dispensed) equals V_0" is true. The latter (antecedent) degree of truth was calculated from the degrees of fulfillment $\sigma_\curlyvee(V_i \preceq V_0)$ of implications

r_k : IF "volume v is V_k" THEN "volume v is V_0",

where one implication r_k, $k \in \{1, 2, 3\}$ was supplied per expert.

Therefore, the FLRpe scheme was applied as described in Algorithm 1. We point out that dispensing stops if and only if at least two volume IN estimates, supplied by two different experts, approximate volume V_0 in an inclusion measure "$\sigma_\curlyvee(.,.) \geq T$" sense for a user-defined threshold T.

V. EXPERIMENTS AND RESULTS

We carried out comparative simulation experiments as described in this section.

A. Disparate Data Representation and Fusion

Recall that the FLRpe scheme here consists of an ensemble of three experts including Expert-1, that is a flowmeter measurement device, Expert-2, that is an ultrasonic level meter device and, Expert-3, that is a human expert supervisor of the industrial dispensing procedure.

First, a dispensed (liquid) volume estimate supplied by Expert-1 was represented by a triangular IN (Fig.9) as follows. Even though our flowmeter device supplies a precise measurement, there is uncertainty regarding the dispensed volume due to both time-delays and the storage capacity of the pipes used to drive a fluid to the mixing tank. The latter uncertainty was modeled by two adjacent uniform pdfs, respectively, one above- and the other below- an obtained flowmeter measurement. For instance, let a flowmeter measurement be either m_1 (Fig.4(a)) or m_2 (Fig.4(b)). The aforementioned two adjacent uniform pdfs are shown in Fig.4(a) as well as Fig.4(b). In conclusion, an estimate for a dispensed liquid volume by Expert-1 had a triangular shape as in Fig.9. The corresponding inclusion measure function $\sigma_\curlyvee(F \preceq V_0)$, for $V_0 = 65$, is plotted in Fig.10 versus the median m.

Second, a dispensed (liquid) volume estimate supplied by Expert-2 was represented by an irregularly shaped IN (Fig.11) as follows. In a short sequence, we obtained a number of $N = 9$ successive measurements of the liquid level in the mixing tank resulting in a population of $N = 9$ estimates of the dispensed liquid volume. In conclusion, from the aforementioned population, we induced a IN (Fig.11) using algorithm CALCIN. The corresponding inclusion measure function $\sigma_\curlyvee(F \preceq V_0)$, for $V_0 = 65$, is plotted in Fig.12 versus the median m.

Third, a dispensed (liquid) volume estimate supplied by Expert-3 was represented by a trapezoidal IN (Fig.13) as follows. A human supervisor of the industrial procedure, based on visual inspection of the transparent tube connected to the side of the mixing tank (Fig.8) as well as based on personal judgement, supplied a numeric estimate m of the middle of an interval $[m - w, m + w]$ which (interval) is the core of a trapezoidal fuzzy set. Furthermore, both trapezoidal tails w_L and w_R in Fig.13 were suggested by Expert-3. Fig.14 displays a typical estimate for a dispensed liquid volume given by Expert-3, where $w = 1$, $w_L = 5$ and $w_R = 2$. The corresponding inclusion measure function $\sigma_\curlyvee(F \preceq V_0)$, for $V_0 = 65$, is plotted in Fig.15 versus the median m.

We remark that both curves in Fig.10 and Fig.15 are smooth because they have been computed analytically using equations in section II-E; whereas, the curve in Fig.12 is not smooth due to the irregularly shaped IN of Fig.11. Furthermore note that, first, the triangular IN (Fig.4) supplied by Expert-1 represents a probability distribution including *a priori* information; in particular, the two adjacent iniform pdfs in either Fig.4(a) or Fig.4(b) represent *a* *priori* information supplied by the user. Second, the irregularly shaped IN (Fig.11) supplied by Expert-2 represents a distribution of measurements and, third, the trapezoidal IN (Fig.14) supplied by Expert-3 represents a fuzzy set. Hence, each expert interprets differently the IN it supplies. In the latter sense, disparate data fussion takes place.

B. Comparative Experimental Results and Discussion

We carried out, comparatively, preliminary computer simulation experiments, using a standard commercial software package (MATLAB), as described in the following.

First, we compared an employment of the mean μ versus the median m of a distribution. Note that a standard practice in the industry is to employ the average/mean value μ of a population of measurements instead of the corresponding median value m as it was demonstrated above (see in section III-B, Example-2). However, the theoretical discussion above (see in the last paragraph of section II-E) has shown that an employment of inequality $m < \mu$, for skewed pdfs, can increase the probability of a dispensed liquid volume "being inside the specifications". In a series of Monte-Carlo computer experiments we confirmed, for both Expert-1 and Expert-2, that a combined employment of m and μ results in fewer violations of the specifications. The latter is significant for our industrial application. Nevertheless, a conceptual problem arises regarding the employment of a median m computed for the fuzzy set supplied by Expert-3 because a median m is meaningless for a fuzzy set. However, due to the one-to-one correspondence between INs and pdfs [34], [35], [39], [40], it follows that for any IN a *median equivalent (parameter)* m can be defined. Moreover, compared with the median m of a pdf, inclusion measure $\sigma_{\curlyvee}(.)$ has the advantage that only $\sigma_{\curlyvee}(.)$ can capture higher-order data statistics; in fact, $\sigma_{\curlyvee}(.)$ can capture all-order data statistics [39], [40], [41].

Second, we comparatively evaluated the performance of our proposed FLRpe scheme. The latter (scheme) was tested in a number of computer simulation experiments assuming a single unreliable expert. More specifically, we assumed that two experts were able to supply accurate (dispensed) liquid volume INs, whereas the third expert supplied a IN either at random or lagging/leading the correct volume. In other words, we used "intact" two of the three inclusion measures $\sigma_{\curlyvee}(F \preceq V_0)$ curves shown in Fig.10, Fig.12 and Fig.15, whereas we used either random samples of the third curve or a left/right-translated version of the third curve. In conclusion, an *alternative decision scheme* has employed the average of the three inclusion measures values supplied by the three experts.

Each one of the three inclusion measures $\sigma_{\curlyvee}(F \preceq V_0)$ curves shown in Fig.10, Fig.12 and Fig.15 was sampled at specific values of the parameter m – Note that successive parameter m samples correspond to successive time instances. Then, both the FLRpe and the aforementioned *alternative decision scheme* were applied at every (data) sampling instance. We confirmed, using threshold $T = 0.93$, that the FLRpe scheme always accurately stops dispensing, whereas the alternative decision scheme may fail even at all (data) sampling instances. Note also that a single expert never performed better than the FLRpe scheme. Such reliable decision-making, as the FLRpe can provide, can be of critical importance in our industrial application due to the fact that one of the three experts may, occasionally, fail as it will be detailed in a future publication.

VI. CONCLUSION

Automated as well as accurate dispensing towards retaining a competitive product quality is of interest in a wide range of industrial applications including plastics, chemicals, dyeing, pharmaceuticals, and foods. This work has demonstrated a novel scheme, namely *Fuzzy Lattice Reasoning pairwise ensemble*, or FLRpe for short, for industrial dispensing based on (FLR) reasoning, which may accommodate imprecision/uncertainty/vagueness in the data. The FLRpe operates by considering, pairwise, all combinations of a number of expert implications based on the sigma-join $\sigma_{\curlyvee}(.,.)$ inclusion measure. Preliminary experimental results have been encouraging.

This work has also presented a formal information fusion framework, namely the Cartesian product lattice $(\mathsf{F}^N, \preceq)$ of Intervals'Numbers (INs), towards an integration of disparate types of data including (intervals of) real numbers as well as probability/possibility distributions. Furthermore, a number of mathematical improvements were presented. Several illustrative examples have demonstrated practical advantages of the proposed techniques including the employment of granular input data as well as the sensible employment of a sparse rule base.

Future plans include, first, a study of implication $p \rightarrow q$ based on both inclusion measures $\sigma_{\curlywedge}(.,.)$ and $\sigma_{\curlyvee}(.,.)$ and, second, an industrial application of the FLRpe scheme for automated ouzo production. The mathematical instruments presented here may also be especially useful for the design of dynamically evolving fuzzy systems [4], as well as for fuzzy regression analysis [8].

APPENDIX

This Appendix summarizes useful notions and tools regarding lattice theory [7], [35], [43], [59] using an improved mathematical notation [31], [37].

A. Mathematical Background

Given a set P, a binary relation $(\preceq)$ in P is called *partial order* if and only if it satisfies the following conditions: $x \preceq x$ (*reflexivity*), $x \preceq y$ and $y \preceq x \Rightarrow x = y$ (*antisymmetry*), and $x \preceq y$ and $y \preceq z \Rightarrow x \preceq z$ (*transitivity*) – We remark that the *antisymmetry* condition may be replaced by the following equivalent condition: $x \preceq y$ and $x \neq y \Rightarrow y \npreceq x$. If both $x \preceq y$ and $x \neq y$ then we write $x \prec y$. A *partially ordered set*, or *poset* for short, is a pair $(P, \preceq)$, where P is a set and $\preceq$ is a partial order relation in P. Note that, in this work, we employ an improved mathematical notation using, first, "curly" symbols $\curlyvee$, $\curlywedge$, $\preceq$, $\prec$, etc. for general poset/lattice elements and, second, "straight" symbols such as $\vee$, $\wedge$, $\leq$, $<$, etc. for real numbers, i.e. elements of the totally-ordered lattice $(\mathsf{R}, \leq)$.

A *lattice* is a poset $(\mathsf{L}, \preceq)$ any two of whose elements $x, y \in \mathsf{L}$ have both a *greatest lower bound*, or *meet* for short, and a *least upper bound*, or *join* for short, denoted by $x \curlywedge y$ and $x \curlyvee y$, respectively. Two elements $x, y \in \mathsf{L}$ in a lattice $(\mathsf{L}, \preceq)$ are called *comparable*, symbolically $x \sim y$, if and only if it is either $x \preceq y$ or $x \succ y$. A lattice $(\mathsf{L}, \preceq)$ is called *totally-ordered* if and only if $x \sim y$ for any $x, y \in \mathsf{L}$. If $x \nsim y$ holds for two elements $x, y \in \mathsf{L}$ of a lattice $(\mathsf{L}, \preceq)$ then x and y are called *incomparable* or, equivalently, *parallel*, symbolically $x||y$.

Given a lattice $(\mathsf{L}, \preceq)$ it is known that $(\mathsf{L}, \preceq^{\partial}) \equiv (\mathsf{L}, \succeq)$ is also a lattice, namely *dual* (lattice), where $\preceq^{\partial}$ denotes the *dual* (i.e. converse) of order relation $\preceq$. Furthermore, it is known that the Cartesian product $(\mathsf{L}_1, \preceq) \times (\mathsf{L}_2, \preceq)$, of two lattices $(\mathsf{L}_1, \preceq)$ and $(\mathsf{L}_2, \preceq)$, is a lattice with order $(x_1, x_2) \preceq (y_1, y_2) \Leftrightarrow x_1 \preceq y_1$ and $x_2 \preceq y_2$. In the latter Cartesian product lattice it holds both $(x_1, x_2) \curlywedge (y_1, y_2) = (x_1 \curlywedge y_1, x_2 \curlywedge y_2)$ and $(x_1, x_2) \curlyvee (y_1, y_2) = (x_1 \curlyvee y_1, x_2 \curlyvee y_2)$. It follows that the Cartesian product $(\mathsf{L}, \succeq) \times (\mathsf{L}, \preceq) \equiv (\mathsf{L} \times \mathsf{L}, \succeq \times \preceq)$ is a lattice with order $(x_1, x_2) \preceq (y_1, y_2) \Leftrightarrow x_1 \succeq y_1$ and $x_2 \preceq y_2$; moreover, $(x_1, x_2) \curlywedge (y_1, y_2) = (x_1 \curlyvee y_1, x_2 \curlywedge y_2)$ and $(x_1, x_2) \curlyvee (y_1, y_2) = (x_1 \curlywedge y_1, x_2 \curlyvee y_2)$. An element of lattice $(\mathsf{L} \times \mathsf{L}, \succeq \times \preceq)$ will be denoted by a pair of L elements within square brackets, e.g. $[a, b]$.

Our interest, here, is in *complete* lattices. Recall that a lattice $(\mathsf{L}, \preceq)$ is called *complete* when each of its subsets X has both a greatest lower bound and a least upper bound in L; hence, for $X = \mathsf{L}$ it follows that a complete lattice has both a *least* and a *greatest* element. In the interest of simplicity, here we use the same symbols O and I to denote the least and the greatest element, respectively, in any complete lattice. Likewise, we use the same symbol $\preceq$ to denote the partial order relation in any (complete) lattice. Consider the following definition.

Definition 4: Let $(\mathsf{L}, \preceq)$ be a complete lattice with least and greatest elements O and I, respectively. An *inclusion measure* in $(\mathsf{L}, \preceq)$ is a function $\sigma : \mathsf{L} \times \mathsf{L} \to [0, 1]$, which satisfies the following conditions

I0. $\sigma(x, O) = 0, \forall x \neq O$.

I1. $\sigma(x, x) = 1, \forall x \in \mathsf{L}$.

I2. $x \curlywedge y \prec x \Rightarrow \sigma(x, y) < 1$.

I3. $u \preceq w \Rightarrow \sigma(x, u) \leq \sigma(x, w)$.

We remark that an inclusion measure $\sigma(x, y)$ can be interpreted as the fuzzy degree to which x is less than y; therefore notation $\sigma(x \preceq y)$ may be used instead of $\sigma(x, y)$.

B. Useful Mathematical Instruments

Two different inclusion measures are presented next, based on a *positive valuation*[3] function.

Theorem 6.1: Let function $v : \mathsf{L} \to \mathsf{R}$ be a positive valuation in a complete lattice $(\mathsf{L}, \preceq)$ such that $v(O) = 0$; then both functions *sigma-meet* $\sigma_{\curlywedge}(x, y) = \frac{v(x \curlywedge y)}{v(x)}$ and *sigma-join* $\sigma_{\curlyvee}(x, y) = \frac{v(y)}{v(x \curlyvee y)}$ are inclusion measures.

Due to practical restrictions, we introduce two constraints on positive valuation functions, next. First, in order to satisfy condition I0 of Definition 4, our interest is in positive valuation functions such that "$v(O) = 0$". Second, since a positive valuation function $v : \mathsf{L} \to \mathsf{R}$ implies a metric (distance) function $d : \mathsf{L} \times \mathsf{L} \to \mathsf{R}^{\geq 0}$ given by $d(a, b) = v(a \curlyvee b) - v(a \curlywedge b)$, furthermore infinite distances between lattice elements are not desired, our second constraint is "$v(I) < +\infty$". Our interest, in the context of this work, focuses solely on inclusion measure functions.

[3]*Positive valuation* in a general lattice $(\mathsf{L}, \preceq)$ is a real function $v : \mathsf{L} \times \mathsf{L} \to \mathsf{R}$ that satisfies both $v(x) + v(y) = v(x \curlywedge y) + v(x \curlyvee y)$ and $x \prec y \Rightarrow v(x) < v(y)$.

A bijective (i.e. one-to-one) *dual isomorphic*[4] function $\theta : \mathsf{L} \to \mathsf{L}$ such that $x \prec y \Leftrightarrow \theta(x) \succ \theta(y)$, in a lattice $(\mathsf{L}, \preceq)$, can be used for extending an inclusion measure from a lattice $(\mathsf{L}, \preceq)$ to the corresponding lattice of intervals. Given a dual isomorphic function $\theta : \mathsf{L} \to \mathsf{L}$ there follow, by definition, both $\theta(x \curlywedge y) = \theta(x) \curlyvee \theta(y)$ and $\theta(x \curlyvee y) = \theta(x) \curlywedge \theta(y)$. The latter equalities are handy in the proof of the following Proposition.

Proposition 6.2: Let real function $v : \mathsf{L} \to \mathsf{R}$ be a *positive valuation* in a lattice $(\mathsf{L}, \preceq)$; moreover, let bijective function $\theta : \mathsf{L} \to \mathsf{L}$ be *dual isomorphic* in $(\mathsf{L}, \preceq)$, i.e. $x \prec y \Leftrightarrow \theta(x) \succ \theta(y)$. Then, function $v_\Delta : \mathsf{L} \times \mathsf{L} \to \mathsf{R}$ given by $v_\Delta(a, b) = v(\theta(a)) + v(b)$ is a positive valuation in lattice $(\mathsf{L} \times \mathsf{L}, \succeq \times \preceq)$.

Proof

1. First, we show that $v_\Delta(a, b) + v_\Delta(c, d) = v_\Delta((a, b) \curlywedge (c, d)) + v_\Delta((a, b) \curlyvee (c, d))$ as follows.
   $v_\Delta(a, b) + v_\Delta(c, d) = [v(\theta(a)) + v(b)] + [v(\theta(c)) + v(d)] = [v(\theta(a)) + v(\theta(c))] + [v(b) + v(d)] = [v(\theta(a) \curlywedge \theta(c)) + v(\theta(a) \curlyvee \theta(c))] + [v(b \curlywedge d) + v(b \curlyvee d)] = [v(\theta(a \curlyvee c)) + v(\theta(a \curlywedge c))] + [v(b \curlywedge d) + v(b \curlyvee d)] = [v(\theta(a \curlyvee c)) + v(b \curlywedge d)] + [v(\theta(a \curlywedge c)) + v(b \curlyvee d)] = v_\Delta(a \curlyvee c, b \curlywedge d) + v_\Delta(a \curlywedge c, b \curlyvee d) = v_\Delta((a, b) \curlywedge (c, d)) + v_\Delta((a, b) \curlyvee (c, d))$.
2. Second, we show that $(a, b) \prec (c, d) \Rightarrow v_\Delta(a, b) < v_\Delta(c, d)$ as follows.
   $(a, b) \prec (c, d) \Rightarrow$ either $(a \succ c$ and $b \preceq d)$ or $(a \succeq c$ and $b \prec d) \Rightarrow$ either $(\theta(a) \prec \theta(c)$ and $b \preceq d)$ or $(\theta(a) \preceq \theta(c)$ and $b \prec d) \Rightarrow$ either $(v(\theta(a)) < v(\theta(c))$ and $v(b) \leq v(d))$ or $(v(\theta(a)) \leq v(\theta(c))$ and $v(b) < v(d)) \Rightarrow v(\theta(a)) + v(b) < v(\theta(c)) + v(d) \Rightarrow v_\Delta(a, b) < v_\Delta(c, d)$.

The latter completes the proof of Proposition 6.2.

We remark that Proposition 6.2 has been proven, quite restrictively, for a totally-ordered lattice $(\mathsf{L}, \preceq)$ in [43].

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[4] A function $\psi : (P, \preceq) \to (Q, \preceq)$, between posets $(P, \preceq)$ and $(Q, \preceq)$, is called *(order) isomorphic* iff both "$x \preceq y \Leftrightarrow \psi(x) \preceq \psi(y)$" and "$\psi$ is onto Q"; then, posets $(P, \preceq)$ and $(Q, \preceq)$ are called *isomorphic*, symbolically $(P, \preceq) \cong (Q, \preceq)$.

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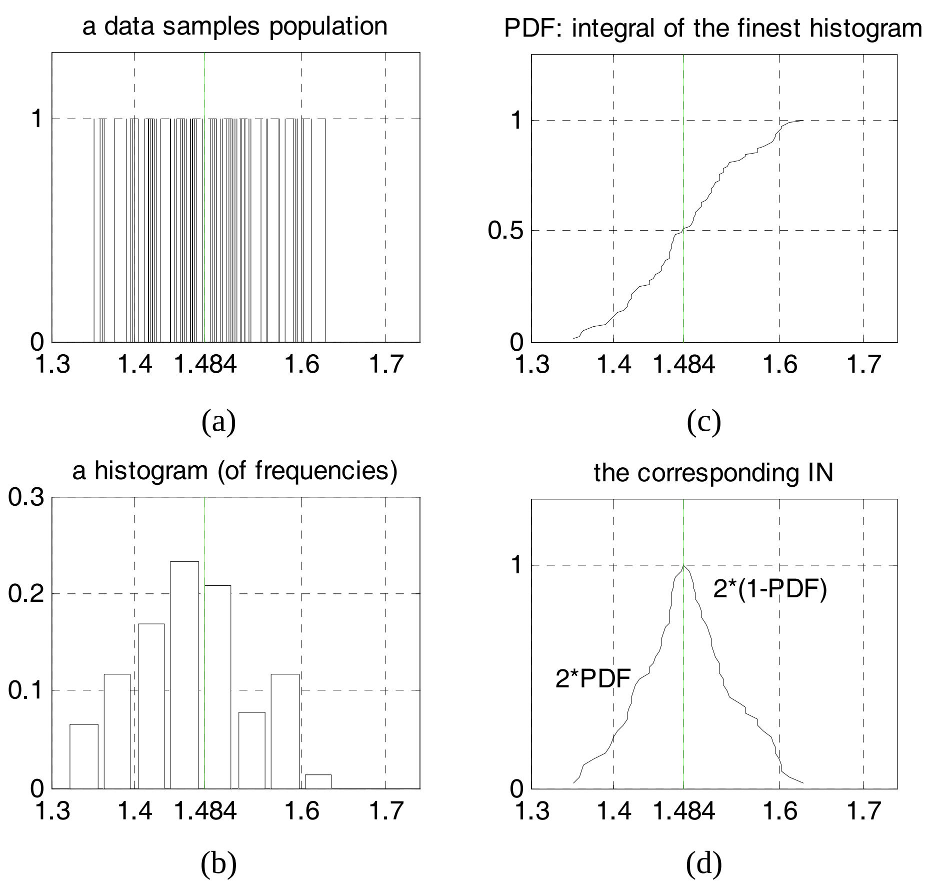


Fig. 1. Calculation of a IN from a population of data samples. (a) The data samples with median $m = 1.484$. (b) A histogram of the data. (c) The corresponding cumulative distribution function (PDF). (d) Computation of a IN from the corresponding PDF; that is, algorithm CALCIN.

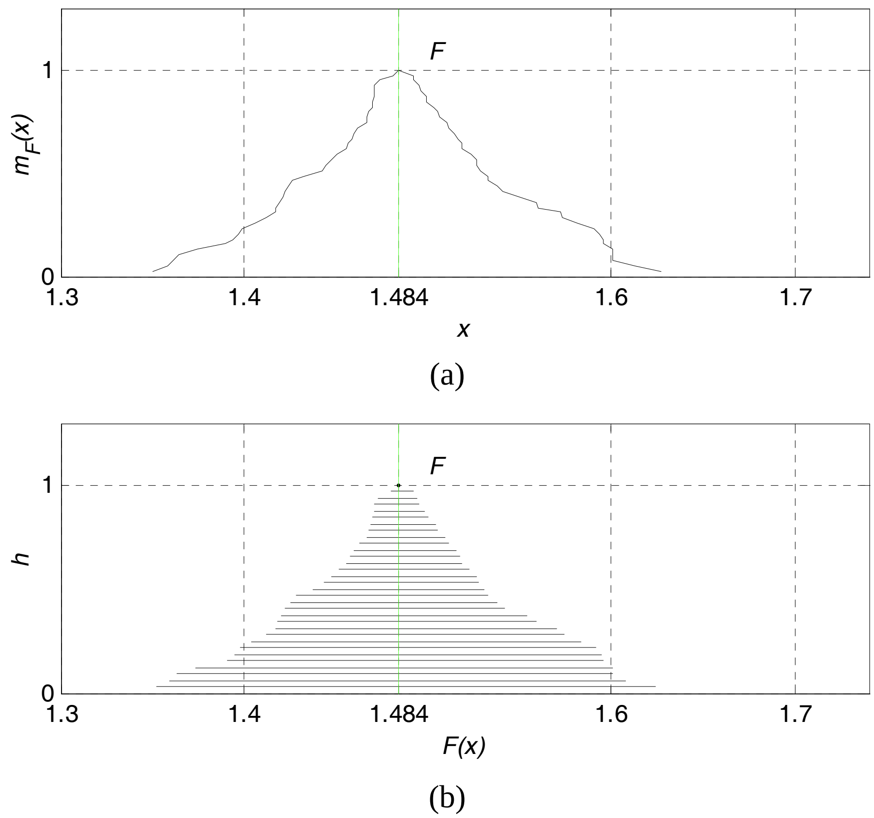


Fig. 2. The two different representations of a IN F from Fig.1(d). (a) The membership-function-representation $m_F(x)$. (b) The interval-representation for $L = 32$ different levels spaced evenly over the interval $(0, 1]$.

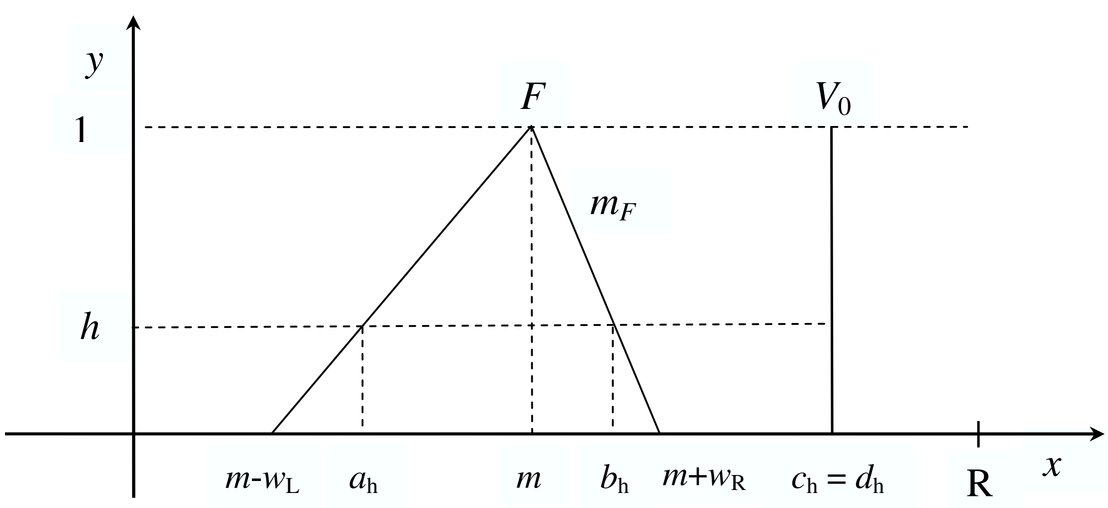


Fig. 3. Two INs including a triangular IN F with membership function m_F, specified by the three numbers $m - w_L$, m, $m + w_R$, and a trivial IN V_0. A horizontal line at height $h \in (0, 1]$ intersects IN F at points a_h and b_h, moreover it intersects trivial IN V_0 at $c_h = d_h = V_0$.

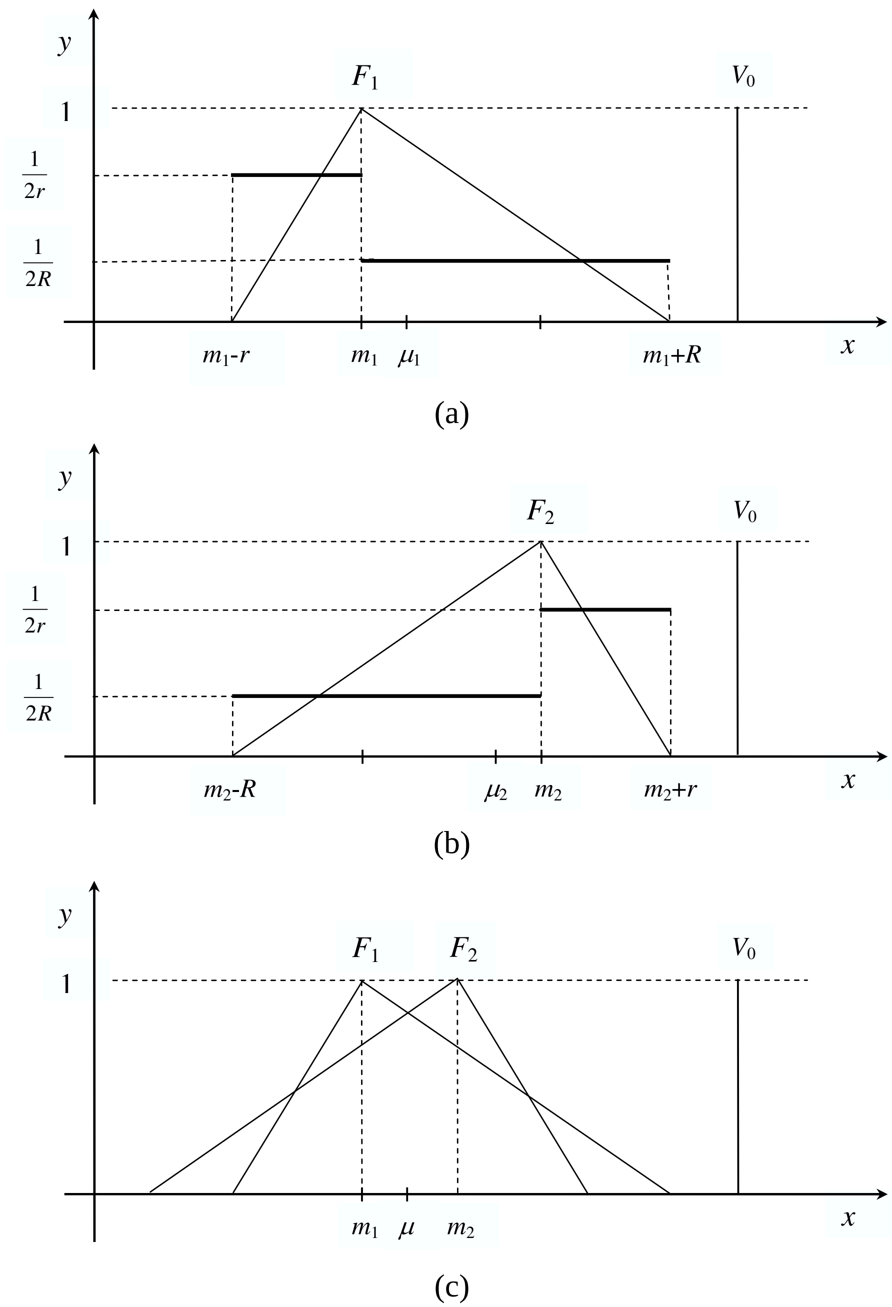


Fig. 4. Triangular INs F_1, F_2 and trivial IN V_0. (a) IN F_1 corresponds to a piecewise-uniform $p_1(x)$ pdf such that $p_1(x) = \frac{1}{2r}$ for $m_1 - r \leq x \leq m_1$, whereas $p_1(x) = \frac{1}{2R}$ for $m_1 \leq x \leq m_1 + R$. (b) IN F_2 corresponds to a piecewise-uniform $p_2(x)$ pdf such that $p_2(x) = \frac{1}{2R}$ for $m_2 - R \leq x \leq m_2$, whereas $p_2(x) = \frac{1}{2r}$ for $m_2 \leq x \leq m_2 + r$. (c) INs F_1 and F_2 were placed so as the corresponding pdfs $p_1(x)$ and $p_2(x)$, respectively, have identical means, i.e. $\mu_1 = \mu = \mu_2$. Note that the standard deviations of $p_1(x)$ and $p_2(x)$ are also identical, i.e. $\sigma_1 = \sigma_2$.

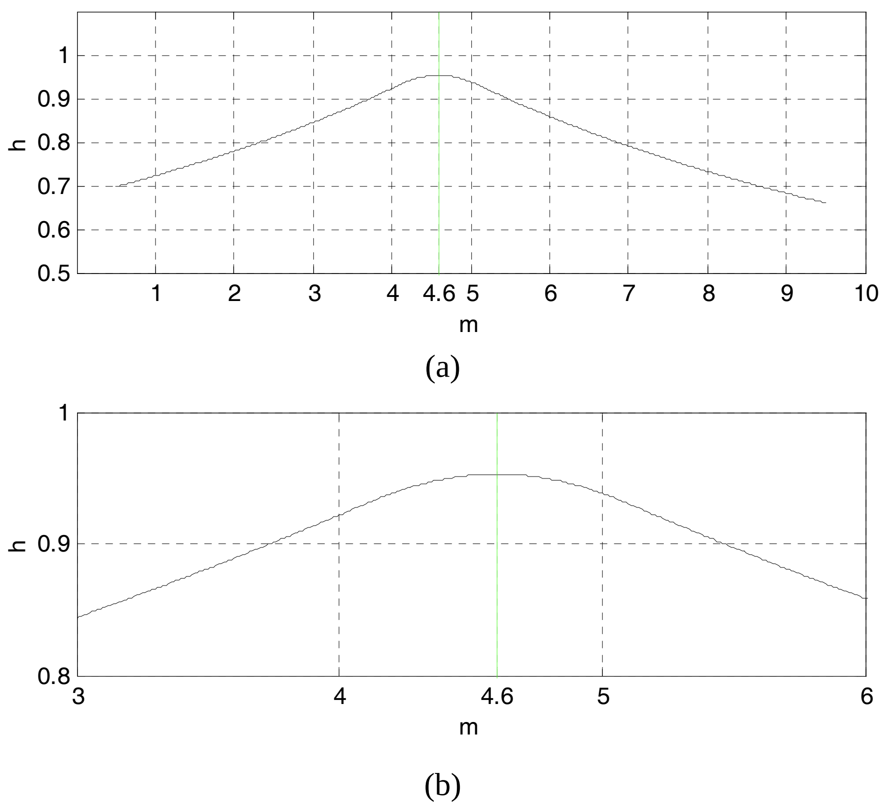
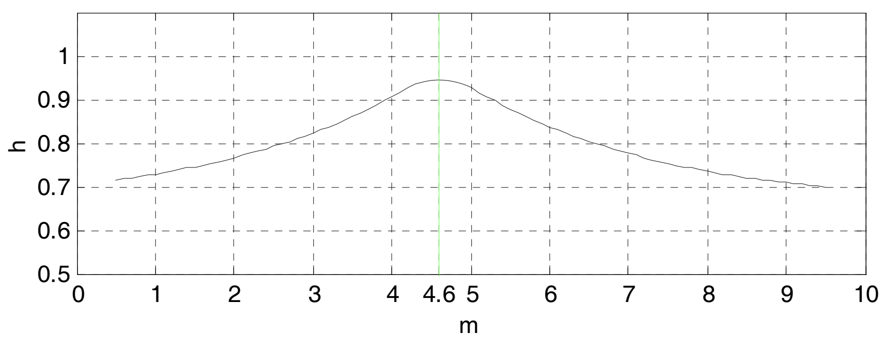
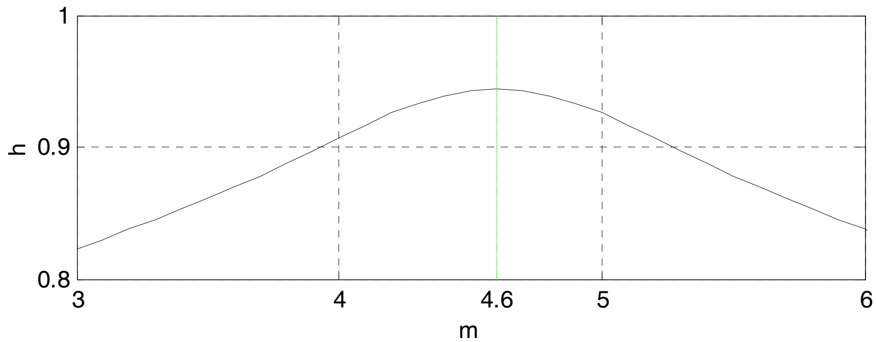


Fig. 5. (a) Inclusion measure $\sigma_{\curlyvee}(F \preceq V_0)$ is plotted versus its median m, where INs F and V_0 are shown in Fig.3, using parameter values $w_L = w_R = 0.5$ and $V_0 = 4.6$; moreover, both the linear positive valuation $v(x) = x$ and the dual isomorphic function $\theta(x) = 10 - x$ were used. (b) The above figure is shown in the vicinity of its global maximum at $m = 4.6$.



(a)



(b)

Fig. 6. (a) Inclusion measure $\sigma_{\curlyvee}(F \preceq V_0)$ is plotted versus its median m, where INs F and V_0 are shown in Fig.3 using parameter values $w_L = w_R = 0.5$ and $V_0 = 4.6$; moreover, both the sigmoid positive valuation $v(x) = \frac{1}{1+e^{-0.5(x-4.6)}}$ and the dual isomorphic function $\theta(x) = 2(4.6) - x$ were used. (b) The above figure is shown in the vicinity of its global maximum at $m = 4.6$.

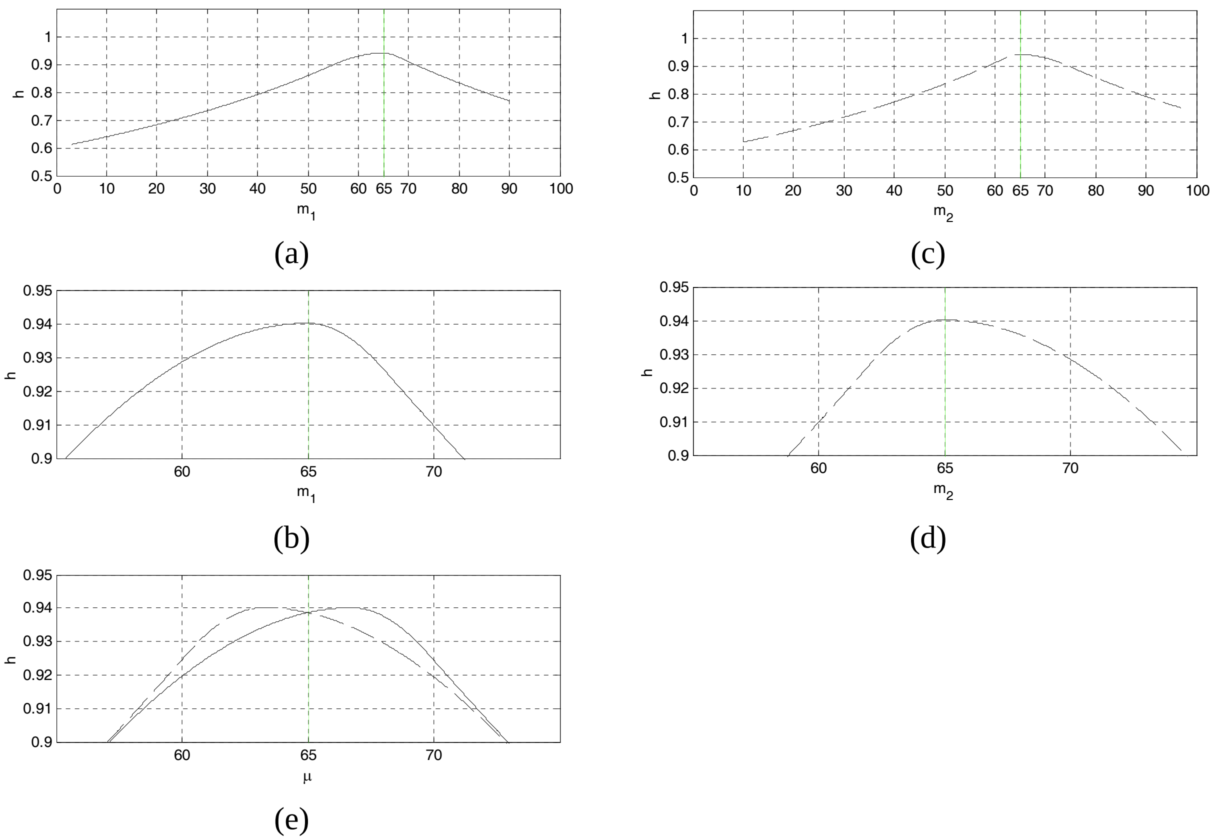


Fig. 7. INs F_1, F_2 and V_0 are shown in Fig.4 with $r = 3$ and $R = 10$, moreover trivial IN V_0 is located at 65. (a) Inclusion measure $\sigma_{\curlyvee}(F_1 \preceq V_0)$ is plotted versus its median m_1. (b) The latter figure is shown in the vicinity of its global maximum at $m_1 = 65$. (c) Inclusion measure $\sigma_{\curlyvee}(F_2 \preceq V_0)$ is plotted versus its median m_2. (d) The latter figure is shown in the vicinity of its global maximum at $m_2 = 65$. (e) Inclusion measures $\sigma_{\curlyvee}(F_1 \preceq V_0)$ and $\sigma_{\curlyvee}(F_2 \preceq V_0)$ are shown, comparatively, in the vicinity of their global maximum versus their identical mean μ.

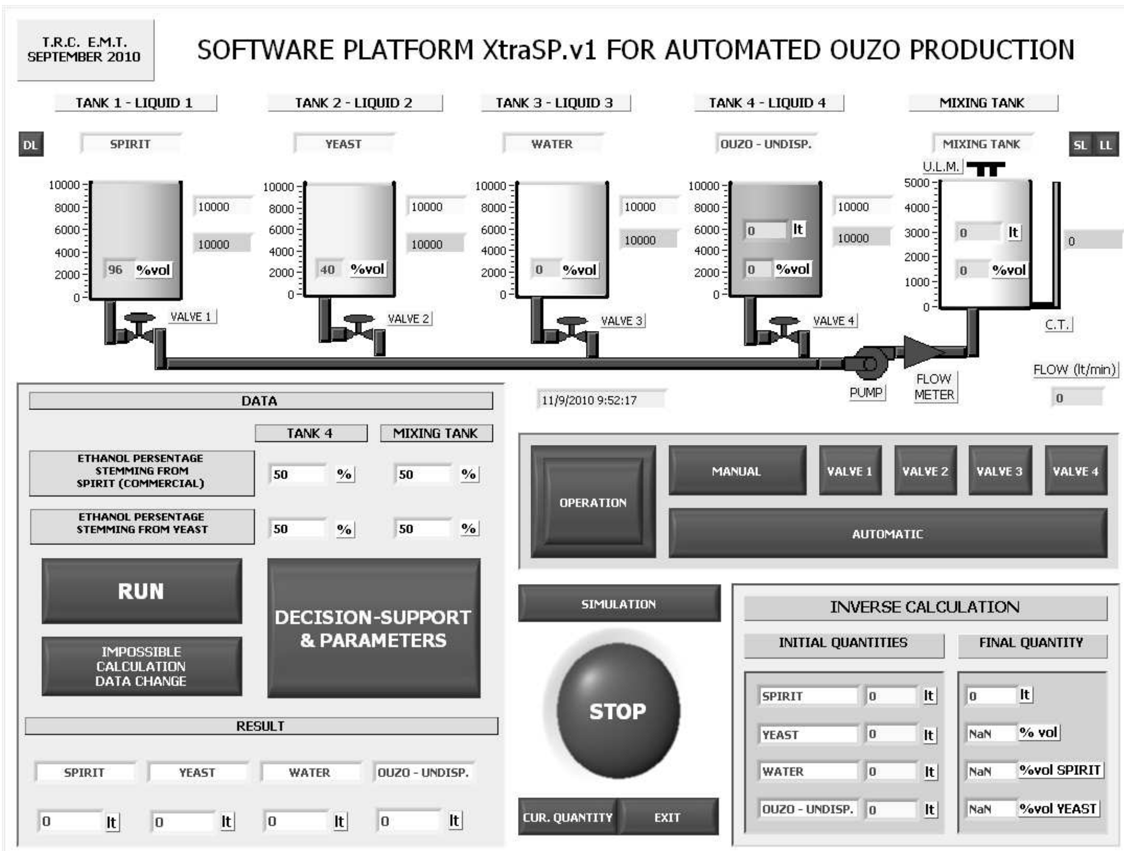
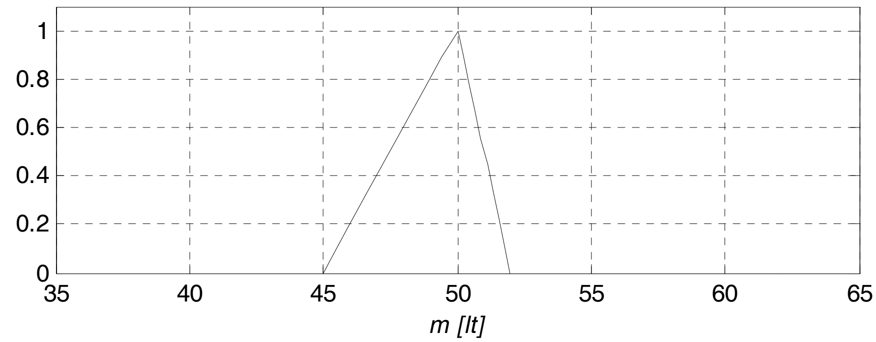
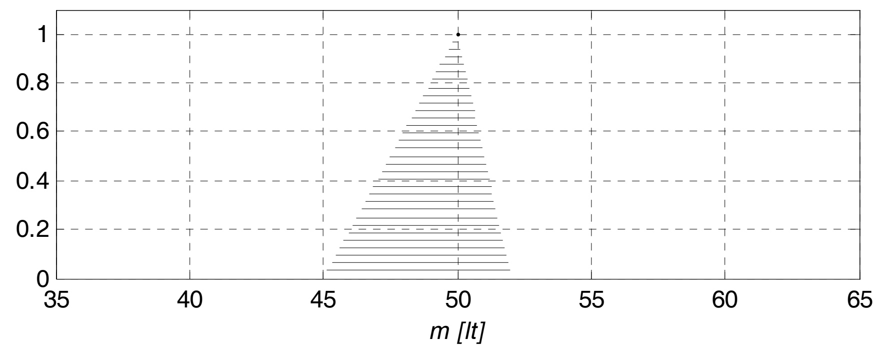


Fig. 8. A fully functional software platform, namely XtraSP.v1, has been developed, in the context of this work, towards an industrial production of *ouzo* (alcoholic) beverage by automating the corresponding liquid dispensing application. Cell label "U.L.M." stands for Ultrasonic Level Meter, moreover cell label "C.T." stands for Communicating Tube.

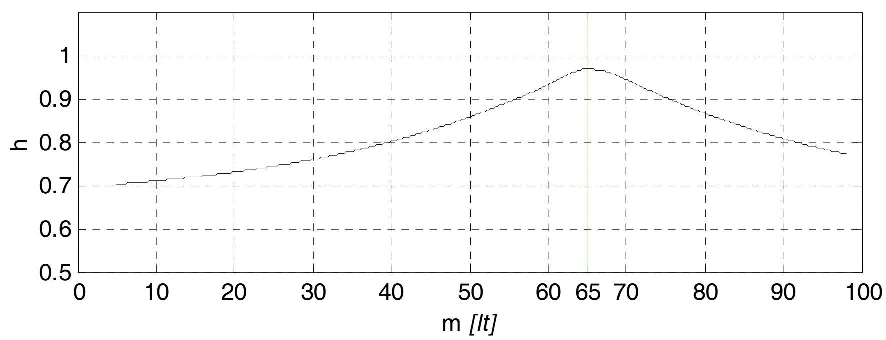


(a)

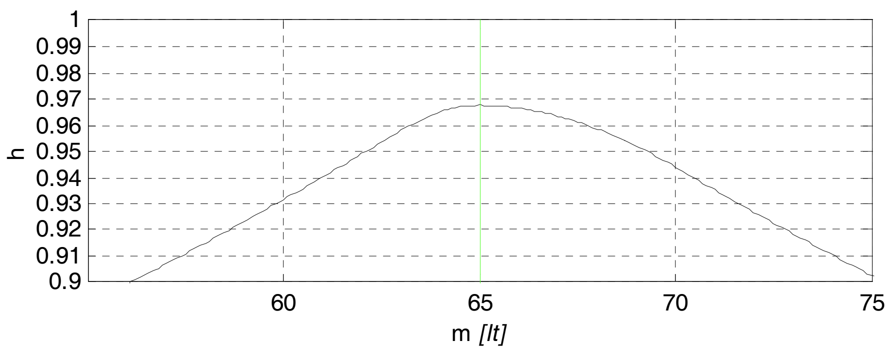


(b)

Fig. 9. Expert-1, that is a flowmeter measurement device, supplied a triangular IN estimate of a dispensed volume as detailed in the text. (a) The membership-function-representation of a dispensed volume estimate. (b) The corresponding interval-representation.



(a)



(b)

Fig. 10. (a) Inclusion measure $\sigma_{\curlyvee}(F \preceq V_0)$ is plotted versus its median m, where IN F is shown in Fig.9, moreover $V_0 = 65$. (b) The above figure is shown in the vicinity of its global maximum at $m = 65$.

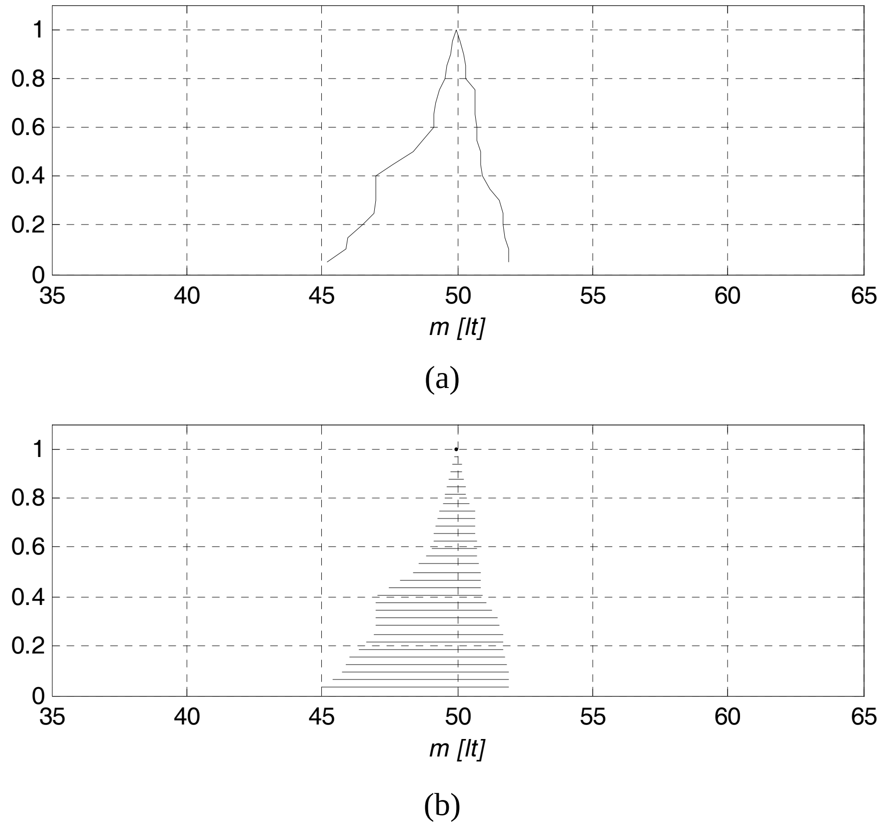


Fig. 11. Expert-2, that is a ultrasonic level meter measurement (U.L.M.) device, supplied a population of measurements resulting in a IN of irregular shape as an estimate of a dispensed volume as detailed in the text. (a) The membership-function-representation of a dispensed volume estimate. (b) The corresponding interval-representation.

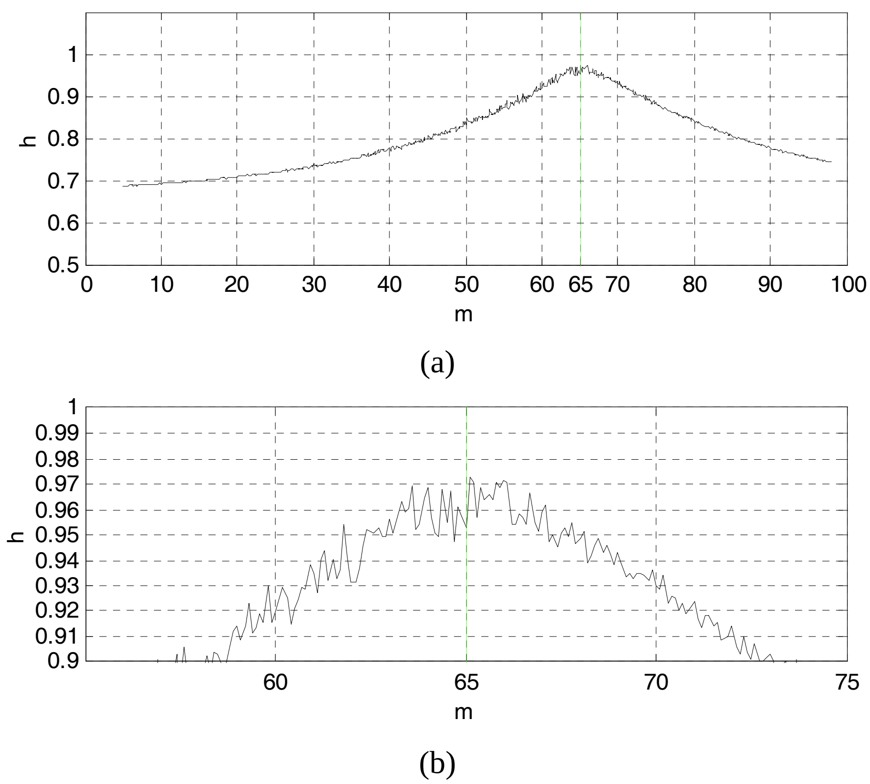


Fig. 12. (a) Inclusion measure $\sigma_{\curlyvee}(F \preceq V_0)$ is plotted versus its median m, where IN F is shown in Fig.11, moreover $V_0 = 65$. (b) The above figure is shown in the vicinity of its global maximum at $m = 65$.

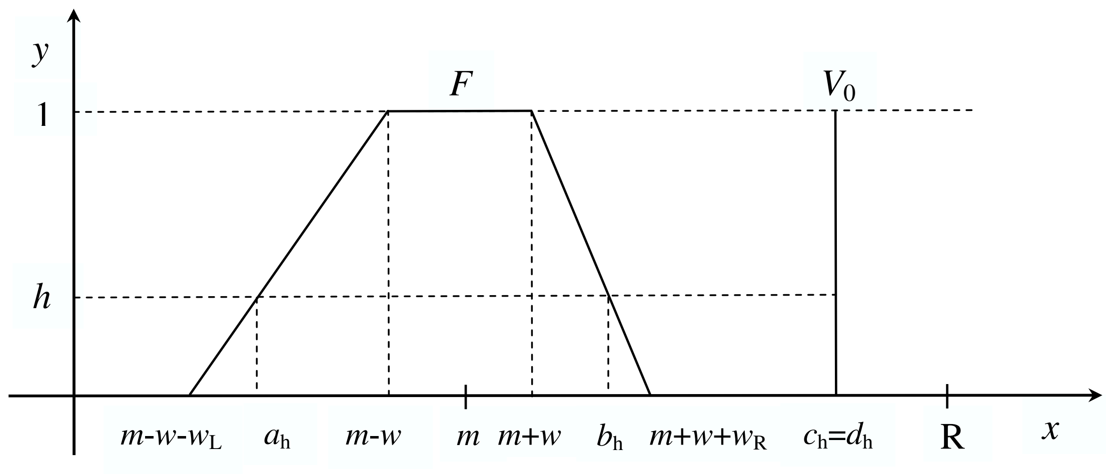


Fig. 13. Two INs including a trapezoidal IN F, specified by the four numbers $m-w-w_L$, $m-w$, $m+w$, $m+w+w_R$ (note that m is the average of numbers $m-w$ and $m+w$), and a trivial IN V_0. A horizontal line at height $h \in (0,1]$ intersects IN F at points a_h and b_h, moreover it intersects trivial IN V_0 at point $c_h = d_h = V_0$.

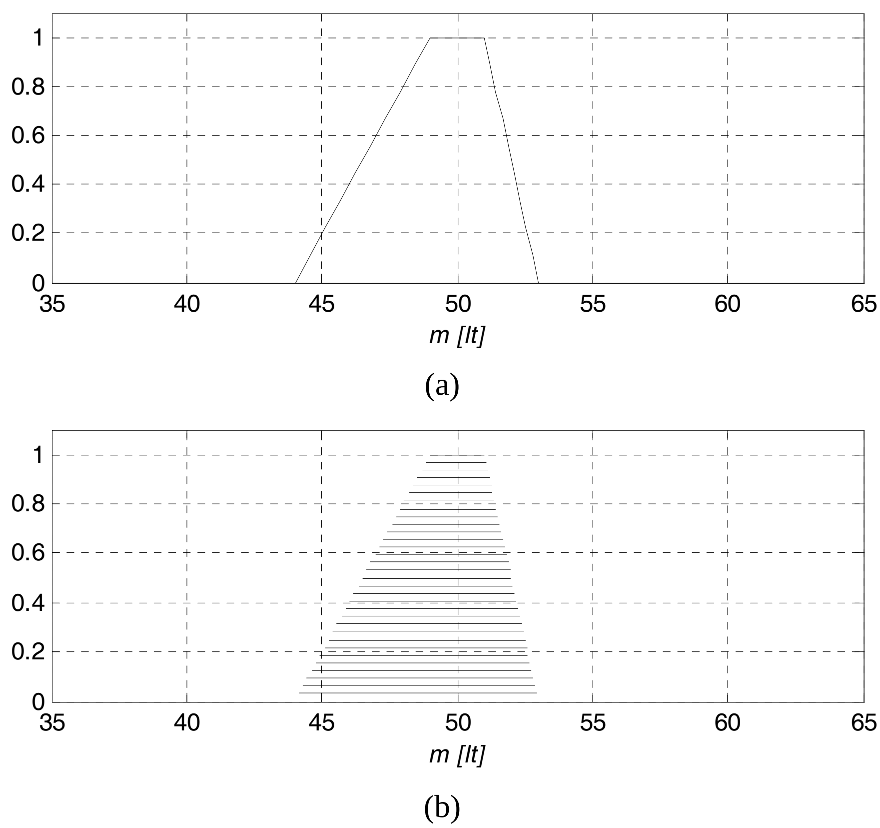
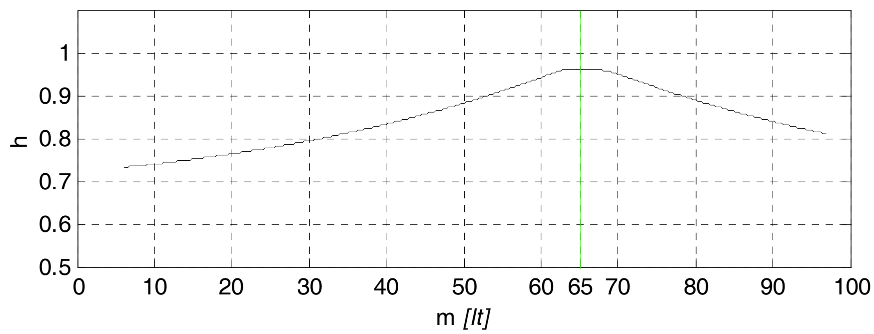
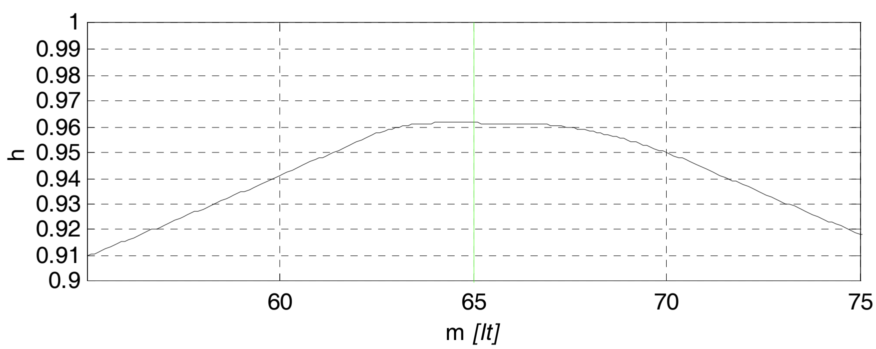


Fig. 14. Expert-3, that is a human expert, supplied a trapezoidal IN estimate of a dispensed volume as detailed in the text. (a) The membership-function-representation of a dispensed volume estimate. (b) The corresponding interval-representation.



(a)



(b)

Fig. 15. (a) Inclusion measure $\sigma_{\curlyvee}(F \preceq V_0)$ is plotted versus its corresponding median parameter m, where IN F is shown in Fig.14, moreover $V_0 = 65$. (b) The above figure is shown in the vicinity of its global maximum at $m = 65$.