A novel distance measure of intuitionistic fuzzy sets and its application to pattern recognition applications

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Abstract

A novel distance measure between two intuitionistic fuzzy sets (IFSs) is proposed in this paper. The introduced measure formulates the information of each set in matrix structure, where matrix norms in conjunction with fuzzy implications can be applied to measure the distance between the IFSs. The advantage of this novel distance measure is its flexibility, which permits different fuzzy implications to be incorporated by extending its applicability to several applications where the most appropriate implication is used. Moreover, the proposed distance might be expressed equivalently by using either intuitionistic fuzzy sets or interval-valued fuzzy sets. Appropriate experimental configurations have taken place in order to compare the proposed distance measure with similar distance measures from the literature, by applying them to several pattern recognition problems. The results are very promising, since the performance of the new distance measure outperforms the corresponding performance of well-known IFSs measures, by recognizing the patterns correctly and with high degree of confidence.

Keywords: Intuitionistic fuzzy sets; interval-valued fuzzy sets; L-fuzzy sets; distance measure; fuzzy implication; pattern recognition; classification.

1. Introduction

Intuitionistic fuzzy sets (IFSs) have been proposed by Atanassov [1-5] as a generalization mathematical framework of the traditional fuzzy sets (FSs) originated from an early work of Zadeh [6,7]. The main advantage of the IFSs is their property to cope with the hesitancy that may exist due to information impression. This is achieved by incorporating a second function, along with the membership function of the conventional FSs, called non-membership function. In this way, apart from the degree of the *belongingness*, the IFSs also combine the notation of the *non-belongingness* in order to better describe the real status of the information.

Since the first introduction of the IFSs and the consequent study on the fundamentals of the IFSs, a lot of attention has been paid on developing distance or similarity measures between the IFSs, as a way to apply them on several problems of the engineering life.

As a result, a lot of measures have been proposed in the past [8-17], each one presenting specific properties and behaviour in real life decision making and pattern recognition application fields.

In this work, an extension of the normalized metric distances suggested in [18] for FSs, on intuitionistic fuzzy sets, based on matrix norms and fuzzy implications, is introduced. It is remarked that there is a strong connection between intuitionistic fuzzy sets (*IFSs*), interval-valued fuzzy sets (*IVFS*) and L-fuzzy sets [19-22]. Also, Wang and He in [22] have proved that the concepts of *IFSs* and intuitionistic L-fuzzy set and the concept of L-fuzzy sets are equivalent. So, the metric distance in *IFSs* that is presented in this paper might be expressed equivalently by using either *IFSs* or *IVFS* or L-fuzzy sets.

Considering the main properties of the resulted distance measure, the measure is used to classify known patterns in several pattern recognition problems, while its performance is compared with that of several methods from the literature, under several experimental configurations.

The paper is organized by presenting some mathematical preliminaries of Intuitionistic Fuzzy Sets Theory, Interval-Valued Fuzzy Sets and fuzzy implications in Section 2. Moreover, Section 3 introduces the novel distance measure between two intuitionistic fuzzy sets using matrix norms and fuzzy implications, while its application on pattern recognition problems and its comparison with other distances is presented in Section 4. Finally, Section 5 summarizes the resulted outcomes of the experimental study and some useful conclusions are drawn.

2. Mathematical Background

2.1 Intuitionistic Fuzzy Sets and Interval-Valued Fuzzy Sets. Basic Notations

Let E denote a universe of discourse. Then a fuzzy set A in E is defined as a set of ordered pairs [6],

$$A=\left\{\left\langle x,\mu_A\left(x\right)\right\rangle / x\in E\right\},$$

where the function $\mu_A:E\rightarrow\left[0,1\right]$, define the degree of membership of the element $x\in E$.

In 1983, Atanassov [1] introduced the concept of the *intuitionistic fuzzy set*, or *IFS* for short, as follows:

An *intuitionistic fuzzy set* A in E is an object of the following form:

$$A=\left\{\left\langle x,\mu_A\left(x\right),v_A\left(x\right)\right\rangle / x\in E\right\}$$

where the functions, $\mu_A:E\rightarrow\left[0,1\right]$ and $v_A:E\rightarrow\left[0,1\right]$, define the degree of membership and the degree of non-membership of the element $x\in E$, respectively and for every $x\in E:0\leq\mu_A\left(x\right)+v_A\left(x\right)\leq 1$.

If $\pi_A\left(x\right)=1-\mu_A\left(x\right)-v_A\left(x\right)$, then $\pi_A\left(x\right)$ is the degree of non-determinacy of the element $x\in E$ to the set A and $\pi_A\left(x\right)\in\left[0,1\right],\forall x\in E$.

It is easily seen that each fuzzy set is a particular case of the intuitionistic fuzzy set. Also, if A is a fuzzy set then $\pi_A\left(x\right)=0,\forall x\in E$

For every two intuitionistic fuzzy sets, A and B several relations and operations are defined (see [2-4]). Here we shall introduce only those which are related to the present research.

(i) $A \subset B \Leftrightarrow \left(\mu_A(x) \le \mu_B(x) \text{ and } v_A(x) \ge v_B(x), \forall x \in E\right)$

(ii) $A \supset B \Leftrightarrow B \subset A$

(iii) $A = B \Leftrightarrow \left(\mu_A(x) = \mu_B(x) \text{ and } v_A(x) = v_B(x), \forall x \in E\right)$

(iv) $\overline{A} = \left\{\left\langle x, v_A(x), \mu_A(x)\right\rangle / x \in E\right\}$

(v) $A \cap B = \left\{\left\langle x, \min\left(\mu_A(x), \mu_B(x)\right), \max\left(v_A(x), v_B(x)\right)\right\rangle / x \in E\right\}$

(vi) $A \cup B = \left\{\left\langle x, \max\left(\mu_A(x), \mu_B(x)\right), \min\left(v_A(x), v_B(x)\right)\right\rangle / x \in E\right\}$

(vii) $A + B = \left\{\left\langle x, \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x), v_A(x)v_B(x)\right\rangle / x \in E\right\}$

(viii) $A \cdot B = \left\{\left\langle x, \mu_A(x)\mu_B(x), v_A(x) + v_B(x) - v_A(x)v_B(x)\right\rangle / x \in E\right\}$

In the following some definitions and some notations used in the interval-valued fuzzy set theory, which are recalled, more or less known in the literature.

In [7] Zadeh introduced the concept of *interval-valued fuzzy sets*, where the degree of membership of an element to a set is characterized not by an element of $[0,1]$ but by a closed subinterval of $[0,1]$.

The interval $[0,1]$ is replaced by the set: $[0,1]^{[2]} = \left\{(a,b) : a,b \in [0,1], a \le b\right\}$. If E is the universal set, then the interval-valued fuzzy sets are mappings

$$A = E \to [0,1]^{[2]}$$

Let A denote an *interval-valued fuzzy set* on E. Then $A(x) = \left[L_A(x), U_A(x)\right] \subseteq [0,1]$, $\forall x \in E$, where L_A, U_A are fuzzy sets that are called the *lower bound* of A and the *upper bound* of A, respectively. When $L_A = U_A$, then A becomes an ordinary fuzzy set.

We recall [1,2,23] that:

An *L-fuzzy set* A on a crisp non-empty set E is a function $A : E \to L$.

An *intuitionistic L-fuzzy set* A in E is an object of the following form:

$$A = \left\{\left\langle x, \mu_A(x), v_A(x)\right\rangle / x \in E\right\},$$

Where $\mu_A : E \to L$ and $v_A : E \to L$ satisfy the condition $\mu_A(x) \le N(v_A(x)), \forall x \in E$, where N is the order-reversing involution on L.

2.2 Metric Distances - Basic Notations

Definition 1: A metric distance d in a set A is a real function $d : A \times A \to R$, which satisfies the following conditions for $x, y, z \in A$:

(i) $d(x, y) = 0 \Leftrightarrow x = y$,

(ii) $d(x, y) = d(y, x)$ (symmetric),

(iii) $d(x, z) + d(z, y) \ge d(x, y)$ (triangle inequality).

Various metric distances, involving fuzzy sets, have been proposed in [24,25]. Some common metrics, which are used for the description of the distance between fuzzy sets, are the following:

If the universe set E is finite, i.e. $E = \{x_1, ..., x_n\}$ then for any two fuzzy subsets A and B of E with membership functions $\mu_A(.)$ and $\mu_B(.)$, respectively, we have:

Hamming distance

$$d_H(A, B) = \sum_{i=1}^{n} \left| \mu_A(x_i) - \mu_B(x_i) \right| \quad (1)$$

Normalized Hamming distance

$$d_{n-H}(A, B) = \frac{1}{n} \sum_{i=1}^{n} \left| \mu_A(x_i) - \mu_B(x_i) \right| \quad (2)$$

Euclidean distance

$$d_E(A, B) = \sqrt{\sum_{i=1}^{n} \left(\mu_A(x_i) - \mu_B(x_i) \right)^2} \quad (3)$$

Normalized Euclidean distance

$$d_{n-E}\left(A,B\right)=\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\mu_A\left(x_i\right)-\mu_B\left(x_i\right)\right)^2} \tag{4}$$

Atanassov suggested [3,8] the following generalization of the above distances Eq.(1)-Eq.(4) for $IFSs$.

Let A, B be $IFSs$ in E, with membership functions $\mu_A(.)$, $\mu_B(.)$ and with non-membership functions $v_A(.), v_B(.)$, respectively, then:

Hamming distance

$$d_H\left(A,B\right)=\frac{1}{2}\sum_{i=1}^{n}\left[\left|\mu_A\left(x_i\right)-\mu_B\left(x_i\right)\right|+\left|\nu_A\left(x_i\right)-\nu_B\left(x_i\right)\right|\right] \tag{5}$$

Normalized Hamming distance

$$d_{n-H}\left(A,B\right)=\frac{1}{2n}\sum_{i=1}^{n}\left[\left|\mu_A\left(x_i\right)-\mu_B\left(x_i\right)\right|+\left|\nu_A\left(x_i\right)-\nu_B\left(x_i\right)\right|\right] \tag{6}$$

Euclidean distance

$$d_E\left(A,B\right)=\sqrt{\frac{1}{2}\sum_{i=1}^{n}\left[\left(\mu_A\left(x_i\right)-\mu_B\left(x_i\right)\right)^2+\left(\nu_A\left(x_i\right)-\nu_B\left(x_i\right)\right)^2\right]} \tag{7}$$

Normalized Euclidean distance

$$d_{n-E}\left(A,B\right)=\sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left[\left(\mu_A\left(x_i\right)-\mu_B\left(x_i\right)\right)^2+\left(\nu_A\left(x_i\right)-\nu_B\left(x_i\right)\right)^2\right]} \tag{8}$$

2.3 Fuzzy Implications - Basic Notations and Definitions

A *fuzzy implication* is a function $\sigma_{\Rightarrow}:\left[0,1\right]\times\left[0,1\right]\rightarrow\left[0,1\right]$, which for any truth values a, $b\in\left[0,1\right]$ of (fuzzy) propositions p, q, respectively, gives the truth value $\sigma_{\Rightarrow}\left(a,b\right)$, of conditional proposition "if p then q". Function $\sigma_{\Rightarrow}(.,.)$ should be an extension of the *classical implication* from the domain $\left\{0,1\right\}$ to the domain $\left[0,1\right]$.

The *implication operator* of classical logic is a map $m:\{0,1\}\times\{0,1\}\rightarrow\{0,1\}$ which satisfies the following conditions: $m(0,0)=m(0,1)=m(1,1)=1$ and $m(1,0)=0$. The latter conditions are typically the minimum requirements for a fuzzy implication operator. In other words, fuzzy implications are required to reduce to the classical implication when truth-values are restricted to 0 and 1; i.e. $\sigma_{\Rightarrow}(0,0)=\sigma_{\Rightarrow}(0,1)=\sigma_{\Rightarrow}(1,1)=1$ and $\sigma_{\Rightarrow}(1,0)=0$.

One way of defining an implication operator *m* in classical logic is using formula $m(a,b)=\bar{a}\vee b$, $a,b\in\{0,1\}$, where $\bar{a}$ denotes the negation of a.

This formula can also be rewritten, based on the law of *absorption of negation* in classical logic, as either

$$m(a,b)=\max\{x\in\{0,1\}:a\wedge x\leq b\},\ a,b\in\{0,1\}$$

or

$$m(a,b)=\bar{a}\vee(a\wedge b)$$

Fuzzy logic extensions of the previous formulas respectively, are

(i) $\sigma_{\Rightarrow}(a,b)=u(n(a),b)$

(ii) $\sigma_{\Rightarrow}(a,b)=\sup\{x\in[0,1]:i(a,x)\leq b\}$

(iii) $\sigma_{\Rightarrow}(a,b)=u(n(a),i(a,b))$

$\forall a,b\in[0,1]$. Where u, i and n denote a fuzzy union, a (continuous) fuzzy intersection, and a fuzzy negation, respectively. Note that functions u and i are dual (with respect to n). Recall that a t-norm i and a t-conorm u are called dual (with respect to a fuzzy negation n) if and only if both $n(i(a,\ b))=u(n(a),n(b))$ and $n(u(a,b))=\ i(n(a),n(b))$ hold $\forall a,b\in[0,1]$.

Fuzzy implications obtained from (i) are usually referred to as *S-implications* (the symbol *S* is often used for denoting *t-conorms*) whereas fuzzy implications obtained from (ii) are called *R-implications*, as they are closely connected with the so-called resituated semi group and fuzzy implications obtained from (iii) are called *QL-implications*, since they were originally employed in quantum logic [26].

A number of basic properties of the classical (logic) implication has been generalized by fuzzy implications. Hence, a number of “reasonable axioms” emerged tentatively for fuzzy implications. Some of the aforementioned axioms are displayed next [26].

A1. $a \le b \Rightarrow \sigma_{\Rightarrow}(a,x) \ge \sigma_{\Rightarrow}(b,x)$	*Monotonicity in first argument*
A2. $a \le b \Rightarrow \sigma_{\Rightarrow}(x,a) \le \sigma_{\Rightarrow}(x,b)$.	*Monotonicity in second argument*
A3. $\sigma_{\Rightarrow}(a,\sigma_{\Rightarrow}(b,x)) = \sigma_{\Rightarrow}(b,\sigma_{\Rightarrow}(a,x))$.	*Exchange property*
A4. $\sigma_{\Rightarrow}(a,b) = \sigma_{\Rightarrow}(n(b),n(a))$	*Contraposition*
A5. $\sigma_{\Rightarrow}(1,b) = b$	*Neutrality of truth.*
A6. $\sigma_{\Rightarrow}(0,a) = 1$	*Dominance of falsity*
A7. $\sigma_{\Rightarrow}(a,a) = 1$	*Identity*
A8. $\sigma_{\Rightarrow}(a,b) = 1 \Leftrightarrow a \le b$	*Boundary Condition*
A9. $\sigma_{\Rightarrow}$ is a continuous function	*Continuity*

Contrary to the above *theoretical* definitions, in fuzzy systems *applications* the predominant practice (known as the *Mamdani* method) is to employ fuzzy *products* (symmetric operators, formally akin to t-norms) instead of implications, and to aggregate the results by union (usually by the $\max\{\ldots\}$). The fuzzy products most commonly used in applications are:

the *Mamdani* rule $\quad \sigma_{\mathrm{M}}(a,b) = \min\{a,b\}$ and

the *Larsen* rule $\quad \sigma_{\mathrm{La}}(a,b) = a \cdot b$.

Fuzzy products clearly do not reduce to the classical implication in the limit. In fact, they differ fundamentally from fuzzy implications in that they: a) abide by “*falsity implies nothing*” (rather than “*everything*”) and b) do not distinguish between predicate (cause) and antecedent (effect).

In view of these properties, the Mamdani method is best suited to inference based on phenomenological information [18]. We refer to σ_{M}, σ_{La} as the "engineering implications", contrary to the fuzzy implications [27].

We refer to and employ four fuzzy implications, two fuzzy products and a novel fuzzy implication stemming from a fuzzy lattice inclusion measure. All these are listed here to avoid repetition:

Reichenbach: $\sigma_{\mathrm{R}}(a,b)=1-a+ab$ (is *S*-implication),

Gödel: $\sigma_{\mathrm{G}}(a,b)=\begin{cases}1, \text{ for } a\le b\\ b, \text{ for } a>b\end{cases}$ (is *R*-implication),

Lukasiewicz: $\sigma_{\mathrm{L}}(a,b)=\min\{1,1-a+b\}$ (is both *S*-implication and *R*-implication),

Kleene-Dienes: $\sigma_{\mathrm{KD}}(a,b)=\max\{1-a,b\}$ (is both *S*-implication and *QL*-implication),

Mamdani: $\sigma_{\mathrm{M}}(a,b)=\min\{a,b\}$,

Larsen: $\sigma_{\mathrm{La}}(a,b)=ab$.

In [28] a novel fuzzy implication $\sigma_T(a,b)=\dfrac{f(b)}{f(a\vee b)}$ has been presented, where $a,b\in[0,1]$ and function $f:[0,1]\to[0,1]$, stemming from a fuzzy lattice inclusion measure function. It was shown that the presented fuzzy implication satisfies a number of "reasonable axioms" and properties of fuzzy implications.

3. A Metric Distance on IFSs

In [18] a new family of normalized metric distances between fuzzy sets based on matrix norms and fuzzy implications is suggested. In this Section, is introduced an extension, of these metric distances, on intuitionistic fuzzy sets (*IFSs*).

Furthermore, it is remarked [29] that if $\Pi_1 = (a_{ij}), \Pi_2 = (b_{ij}), i = 1,\ldots,n,\ j = 1,\ldots,n$ are square matrices then the norm $\| \ \|$ can be used to define a metric d as:

$$d(\Pi_1, \Pi_2) = \|\Pi_1 - \Pi_2\| \tag{9}$$

Let A be IFS in a finite universe $E = \{x_1, ..., x_n\}$, with membership functions $\mu_A(.)$, and with non-membership functions $v_A(.)$, respectively. Let $\sigma_{\Rightarrow}$ be a fuzzy implication. We define the $n \times n$ matrices $\Pi(\mu_A)$ and $\Pi(v_A)$ of $\sigma_{\Rightarrow}$ as follows:

$$\Pi(\mu_A) \triangleq \left[\underset{i=1,\ldots,n}{\sigma_{\Rightarrow}}(\mu_A(x_i), \mu_A(x_i))\right] = \sigma_{\Rightarrow}\left(\begin{bmatrix}\mu_A(x_1)\\ \vdots \\ \mu_A(x_n)\end{bmatrix}, [\mu_A(x_1), \ldots, \mu_A(x_n)]\right)$$

$$= \begin{bmatrix} \sigma_{\Rightarrow}(\mu_A(x_1), \mu_A(x_1)) & \ldots & \ldots & \sigma_{\Rightarrow}(\mu_A(x_1), \mu_A(x_n)) \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{\Rightarrow}(\mu_A(x_n), \mu_A(x_1)) & \ldots & \ldots & \sigma_{\Rightarrow}(\mu_A(x_n), \mu_A(x_n)) \end{bmatrix} \quad \text{and}$$

$$\Pi(v_A) \triangleq \left[\underset{i=1,\ldots,n}{\sigma_{\Rightarrow}}(v_A(x_i), v_A(x_i))\right] = \sigma_{\Rightarrow}\left(\begin{bmatrix}v_A(x_1)\\ \vdots \\ v_A(x_n)\end{bmatrix}, [v_A(x_1), \ldots, v_A(x_n)]\right)$$

$$= \begin{bmatrix} \sigma_{\Rightarrow}(v_A(x_1), v_A(x_1)) & \ldots & \ldots & \sigma_{\Rightarrow}(v_A(x_1), v_A(x_n)) \\ \vdots & \vdots & \vdots & \vdots \\ \sigma_{\Rightarrow}(v_A(x_n), v_A(x_1)) & \ldots & \ldots & \sigma_{\Rightarrow}(v_A(x_n), v_A(x_n)) \end{bmatrix}, \text{ respectively.}$$

Let E denote a universe of discourse, where E is a finite and let Σ_{IFSs}^{E} denote the set of all $IFSs$ in E.

Definition 2: Given two intuitionistic fuzzy sets, $A=\left\{\left\langle x,\mu_A\left(x\right),v_A\left(x\right)\right\rangle / x\in E\right\}$ and $B=\left\{\left\langle x,\mu_B\left(x\right),v_B\left(x\right)\right\rangle / x\in E\right\}$, where $E=\left\{x_1,...,x_n\right\}$ is a finite universe. Also, let $\sigma_{\Rightarrow}$ be a fuzzy implication and any tensor-or operator-norm $\left\|\cdot\right\|$. Then

$$d(A,B;\sigma_{\Rightarrow})\triangleq\left\|\Pi\left(\mu_A\right)-\Pi\left(\mu_B\right)\right\|+\left\|\Pi\left(v_A\right)-\Pi\left(v_B\right)\right\| \tag{10}$$

where $\Pi\left(\mu_.\right)=\left[\underset{i=1,...,n}{\sigma_{\Rightarrow}}\left(\mu_.\left(x_i\right),\mu_.\left(x_i\right)\right)\right]$, $\Pi\left(v_.\right)=\left[\underset{i=1,...,n}{\sigma_{\Rightarrow}}\left(v_.\left(x_i\right),v_.\left(x_i\right)\right)\right]$, defines a metric distance $d:\Sigma_{IFSs}^{E}\times\Sigma_{IFSs}^{E}\rightarrow\left[0,+\infty\right)$.

The above function $d(A,B;\sigma_{\Rightarrow})$ is a metric [18]. So, this definition actually introduces multiple families of metrics with different meanings, according to the binary operator chosen.

In Eq.(10) the norm $\left\|\Pi\right\|$ is computed by using the largest non negative eigenvalue of the positive definite Hermitian matrix $\Pi^T\Pi$ (Π^T is the transpose of matrix Π) [29],

$$\left\|\Pi\right\|=\sqrt{\lambda_{\max}} \tag{11}$$

4. Application to Pattern Recognition Problems

In order to study the ability of the proposed metric to count the distance between two intuitionistic fuzzy sets, a set of experiments have been conducted. For this purpose, four well known from the literature problems, according to which a test sample has to be recognized by classifying it to a specific category, are selected.

In the following examples, attributes correspond to the measurements that are used to describe each class, while the classes are represented by specific patterns that they describe the classes' centroids.

This procedure constitutes the main operation of the minimum-distance classifier, where the test sample is assigned to the class from which its distance is minimum and is described by the following equation:

$$k^* = \arg\min_k \left\{ Dist\left(P_k, S\right) \right\} \tag{12}$$

For comparison reasons some distance and similarity measures from the literature [9,11,14], have been implemented and their ability to recognize the identity of a test sample, by classifying it to the appropriate class, is studied in the following sections.

In order to compare the distance and similarity (in this case the similarity measure is transformed to a distance by using the formula *d=1-S*) measures, a new performance index called *Degree of Confidence (DoC)* is introduced. This factor measures the confidence of each distance metric in recognizing a specific sample that belongs to the pattern (i) and has the following form:

$$DoC^{(i)} = \sum_{i=1, i \neq j}^{n} \left| dist\left(P_j, S\right) - dist\left(P_i, S\right) \right| \tag{13}$$

It is obvious from the above Eq.(13) that the greater $DoC^{(i)}$ the more confident the result of the specific distance metric is. This factor is used in the next experimental sections in order to give a more accurate measurement of the distances' behaviour along with the absolute recognition rate.

Moreover, in the following comparison results the proposed distance measure is denoted in conjunction with the implication type being used. In this way there are seven different distance metrics: *Proposed-T (with σ_T implication [28])*, *Proposed-R (with "Reichenbach" implication)*, *Proposed-G (with "Gödel" implication)*, *Proposed-L (with "Lukasiewicz" implication)*, *Proposed-KD (with "Kleene-Dienes" implication)*, *Proposed-M (with "Mamdani" "engineering implication")* and *Proposed-LR (with "Larsen" "engineering implication")*.

Example 1

This example has been introduced in [14] and corresponds to a pattern recognition problem of 4 classes and 12 attributes, described by the patterns P_1, P_2, P_3, P_4 and the test sample S, as presented in the following Table 1.

Table 1. 4-class/12-attributes problem [14], patterns and test sample.

		Attributes											
		x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}
Pattern #1	$\mu_{P_1}(x)$	0.173	0.102	0.530	0.965	0.420	0.008	0.331	1.000	0.215	0.432	0.750	0.432
	$\nu_{P_1}(x)$	0.524	0.818	0.326	0.008	0.351	0.956	0.512	0.000	0.625	0.534	0.126	0.432
Pattern #2	$\mu_{P_2}(x)$	0.510	0.627	1.000	0.125	0.026	0.732	0.556	0.650	1.000	0.145	0.047	0.760
	$\nu_{P_2}(x)$	0.365	0.125	0.000	0.648	0.823	0.153	0.303	0.267	0.000	0.762	0.923	0.231
Pattern #3	$\mu_{P_3}(x)$	0.495	0.603	0.987	0.073	0.037	0.690	0.147	0.213	0.501	1.000	0.324	0.045
	$\nu_{P_3}(x)$	0.387	0.298	0.006	0.849	0.923	0.268	0.812	0.653	0.284	0.000	0.483	0.912
Pattern #4	$\mu_{P_4}(x)$	1.000	1.000	0.857	0.734	0.021	0.076	0.152	0.113	0.489	1.000	0.386	0.028
	$\nu_{P_4}(x)$	0.000	0.000	0.123	0.158	0.896	0.912	0.712	0.756	0.389	0.000	0.485	0.912
Test Sample	$\mu_S(x)$	0.978	0.980	0.798	0.693	0.051	0.123	0.152	0.113	0.494	0.987	0.376	0.012
	$\nu_S(x)$	0.003	0.012	0.132	0.213	0.876	0.756	0.721	0.732	0.368	0.000	0.423	0.897

For this example, it is prior known that the test sample belongs to class 4 and thus the distances have to take minimum values when the sample compared with the fourth pattern. Table 2, summarizes the distance measures' results along with the degree of confidence of each one. In this table the minimum distance and the three best distances with the highest degree of confidence, have been noted in bold.

Table 2. Distance measures' results.

Distances	Results				
	dist(P_1,S)	dist(P_2,S)	dist(P_3,S)	dist(P_4,S)	$DoC^{(4)}$
d_1 [14]	0.4537	0.4599	0.2107	**0.0338**	1.0230
d_2^1 [14]	0.4311	0.4362	0.1982	**0.0270**	0.9843
1-S_d^1 [9]	0.4311	0.4341	0.1969	**0.0250**	0.9872
1-S_e^1 [11]	0.4311	0.4362	0.1982	**0.0270**	0.9843
1-S_s^1 [11]	0.4311	0.4345	0.1972	**0.0256**	0.9861
1-S_h^1 [11]	0.0756	0.0767	0.0351	**0.0056**	0.1705
Proposed – T	11.2121	10.3332	6.3613	**2.3891**	**20.7393**
Proposed – R	7.1715	7.1012	4.4393	**0.7701**	16.4017
Proposed – G	11.7043	11.0144	6.9669	**2.6149**	**21.8410**
Proposed – L	7.6459	7.2389	4.7612	**0.8205**	17.1844
Proposed – KD	7.5153	7.3560	4.6778	**0.7711**	**17.2357**
Proposed – M	6.6799	6.9323	4.7911	**0.6661**	16.4050
Proposed - LR	6.4985	6.3633	4.6090	**0.8900**	14.8008

A careful study of the above table leads to the conclusion that while all the distances under comparison recognize correctly the test sample, the proposed distance that used the *Gödel* implication (*Proposed-G*) is highly confident. Moreover, the next two distances with the highest degree of confidence are the proposed one, when σ_T and *Kleene-Dienes* implications are used.

Example 2

This example has been introduced in [9] and corresponds to a pattern recognition problem of 3 classes and 3 attributes, described by the patterns P_1, P_2, P_3 and the test sample S, as presented in the following Table 3.

Table 3. 3-class/3-attributes problem [9], patterns and test sample.

		Attributes		
		x_1	**x_2**	**x_3**
Pattern #1	$\mu_{P_1}(x)$	1.0	0.8	0.7
	$\nu_{P_1}(x)$	0.0	0.0	0.1
Pattern #2	$\mu_{P_2}(x)$	0.8	1.0	0.9
	$\nu_{P_2}(x)$	0.1	0.0	0.0
Pattern #3	$\mu_{P_3}(x)$	0.6	0.8	1.0
	$\nu_{P_3}(x)$	0.2	0.0	0.0
Test Sample	$\mu_S(x)$	0.5	0.6	0.8
	$\nu_S(x)$	0.3	0.2	0.1

For this example, it is prior known that the test sample belongs to class 3. Table 4, summarizes the distance measures' results along with the degree of confidence of each one.

Table 4. Distance measures' results.

Distances	**Results**			
	dist(P_1,S)	**dist(P_2,S)**	**dist(P_3,S)**	$DoC^{(3)}$
d_1 [14]	0.2417	0.2417	**0.1583**	0.1667
d_2^1 [14]	0.2167	0.2167	**0.1500**	0.1333
1-S_d^1 [9]	0.2167	0.2167	**0.1500**	0.1333
1-S_e^1 [11]	0.2167	0.2167	**0.1500**	0.1333
1-S_s^1 [11]	0.2167	0.2167	**0.1500**	0.1333
1-S_h^1 [11]	0.1611	0.1611	**0.1056**	0.1111
Proposed – T	1.9283	1.1574	**0.8817**	**1.3223**
Proposed – R	1.1179	0.9185	**0.6029**	0.8306
Proposed – G	2.4367	1.4463	**1.1346**	**1.6137**
Proposed – L	0.6581	0.4348	**0.2414**	0.6100
Proposed – KD	1.5732	1.4028	**0.9439**	**1.0882**
Proposed – M	1.2401	1.3391	**0.8743**	0.8305
Proposed - LR	1.2754	1.4150	**0.8719**	0.9467

The results are common with the previous example, since the proposed distance measure using the *Gödel* implication (*Proposed-G*) outperforms the other distances, totally. The σ_T and *Kleene-Dienes* implications have the same behaviour presenting the second and third better performance among all distances under comparison.

Example 3

This example has been introduced in [12,17] and corresponds to a pattern recognition problem of 3 classes and 3 attributes, described by the patterns P_1, P_2, P_3 and the test sample S, as presented in the following Table 5.

Table 5. 3-class/3-attributes problem [12,17], patterns and test sample.

		Attributes		
		x₁	**x₂**	**x₃**
Pattern #1	$\mu_{P_1}(x)$	0.3	0.2	0.1
	$\nu_{P_1}(x)$	0.3	0.2	0.1
Pattern #2	$\mu_{P_2}(x)$	0.2	0.2	0.2
	$\nu_{P_2}(x)$	0.2	0.2	0.2
Pattern #3	$\mu_{P_3}(x)$	0.4	0.4	0.4
	$\nu_{P_3}(x)$	0.4	0.4	0.4
Test Sample	$\mu_S(x)$	0.3	0.2	0.1
	$\nu_S(x)$	0.3	0.2	0.1

For this example, it is prior known that the test sample belongs to class 1. Table 6, summarizes the distance measures' results along with the degree of confidence of each one.

Table 6. Distance measures' results.

Distances	Results			
	dist(P_1,S)	dist(P_2,S)	dist(P_3,S)	$DoC^{(1)}$
d_1 [14]	**0**	0.0667	0.2000	0.2667
d_2^1 [14]	**0**	0.0667	0.2000	0.2667
1-S_d^1 [9]	0	0	0	0
1-S_e^1 [11]	**0**	0.0667	0.2000	0.2667
1-S_s^1 [11]	**0**	0.0333	0.1000	0.1333
1-S_h^1 [11]	**0**	0.0444	0.1333	0.1778
Proposed – T	**0**	1.7544	1.7544	**3.5087**
Proposed – R	**0**	0.3941	0.6218	1.0159
Proposed – G	**0**	2.8291	2.8291	**5.6582**
Proposed – L	**0**	0.4828	0.4828	0.9657
Proposed – KD	**0**	0.4899	1.2961	1.7860
Proposed – M	**0**	0.3759	1.5033	**1.8792**
Proposed - LR	**0**	0.1200	0.7324	0.8524

The same results and conclusions are drawn from the above Table 6, where the proposed distance measure using the *Gödel* implication (*Proposed-G*) performs better, since it recognizes correctly the test sample but most of all its decision is high confident. Moreover, σ_T and *Kleene-Dienes* implications still present the second and third better performance.

Example 4

This example has been introduced in [12,17] and corresponds to a pattern recognition problem of 3 classes and 3 attributes, described by the patterns P_1, P_2, P_3 and the test sample S, as presented in the following Table 7.

Table 7. 3-class/3-attributes problem [12,17], patterns and test sample.

		Attributes		
		x_1	x_2	x_3
Pattern #1	$\mu_{P_1}(x)$	0.1	0.5	0.1
	$\nu_{P_1}(x)$	0.1	0.1	0.9
Pattern #2	$\mu_{P_2}(x)$	0.5	0.7	0.0
	$\nu_{P_2}(x)$	0.5	0.3	0.8
Pattern #3	$\mu_{P_3}(x)$	0.7	0.1	0.4
	$\nu_{P_3}(x)$	0.2	0.8	0.4
Test Sample	$\mu_S(x)$	0.4	0.6	0.0
	$\nu_S(x)$	0.4	0.2	0.8

For this example, it is prior known that the test sample belongs to class 2. Table 8, summarizes the distance measures' results along with the degree of confidence of each one.

Table 8. Distance measures' results.

Distances	Results			
	dist(P_1,S)	dist(P_2,S)	dist(P_3,S)	$DoC^{(2)}$
d_1 [14]	0.1667	**0.0667**	0.4167	0.4500
d_2^1 [14]	0.1667	**0.0667**	0.4000	0.4333
1-S_d^1 [9]	1.1102x10^{16}	**0**	0.4000	0.4000
1-S_e^1 [11]	0.1667	**0.0667**	0.4000	0.4333
1-S_s^1 [11]	0.0833	**0.0333**	0.4000	0.4167
1-S_h^1 [11]	0.1111	**0.0444**	0.2778	0.3000
Proposed – T	1.5637	**0.2401**	2.2564	**3.3400**
Proposed – R	0.8476	**0.2749**	1.4981	1.7959
Proposed – G	1.8127	**0.2618**	2.5306	**3.8197**
Proposed – L	0.9153	**0.2828**	1.4762	1.8258
Proposed – KD	1.0883	**0.4449**	1.7905	**1.9888**
Proposed – M	1.0523	**0.4732**	1.4065	1.5125
Proposed - LR	0.6291	**0.4263**	1.1783	0.9547

Also, in this example the results remain the same as in the previous cases.The *Gödel* implication (*Proposed-G*) can better distinguish the patterns by giving confident decision about the belongingness of the test sample to class 2. Moreover, σ_T and *Kleene-Dienes* implications still present the second and third better performance.

Conclusively, the proposed distance measure gives the same recognition rates with the other distances from the literature, but it results to more confident decisions. This high confidence nature of this novel distance metric makes it appropriate to difficult pattern recognition problems where there is significant information hesitancy.

5. Conclusion

A novel distance metric between intuitionistic fuzzy sets, which is making use of matrix norms and fuzzy implications, was proposed in the previous sections. The introduced distance is very flexible in the sense that enables the usage of an appropriate fuzzy implication regarding the application, by resulting to a wide range of distances of different properties and capabilities.

The above study has shown that the *Gödel* implication (*Proposed-G*) performs not only better than the older distance measures from the literature but also better than the same distance with different implication types, as compared to several pattern recognition problems.

The present work constitutes a first study of such type of distance measures based on fuzzy implications and future research on several real pattern recognition problems is needed in order to establish the proposed methodology as a concrete pattern classification framework.

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