Neural/Fuzzy Computing Based on Lattice Theory

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INTRODUCTION

Computational Intelligence (CI) consists of an evolving collection of methodologies often inspired from nature (Bonissone, Chen, Goebel & Khedkar, 1999, Fogel, 1999, Pedrycz, 1998). Two popular methodologies of CI include neural networks and fuzzy systems.

Lately, a unification was proposed in CI, at a "data level", based on lattice theory (Kaburlasos, 2006). More specifically, it was shown that several types of data including vectors of (fuzzy) numbers, (fuzzy) sets, 1D/2D (real) functions, graphs/trees, (strings of) symbols, etc. are partially(lattice)-ordered. In conclusion, a unified cross-fertilization was proposed for knowledge representation and modeling based on lattice theory with emphasis on clustering, classification, and regression applications (Kaburlasos, 2006).

Of particular interest in practice is the totally-ordered lattice $(\mathsf{R},\leq)$ of real numbers, which has emerged historically from the conventional measurement process of successive comparisons. It is known that $(\mathsf{R},\leq)$ gives rise to a hierarchy of lattices including the lattice $(\mathsf{F},\leq)$ of *fuzzy interval numbers*, or FINs for short (Papadakis & Kaburlasos, 2007).

This article shows extensions of two popular neural networks, i.e. *fuzzy-ARTMAP* (Carpenter, Grossberg, Markuzon, Reynolds & Rosen 1992) and *self-organizing map* (Kohonen, 1995), as well as an extension of conventional *fuzzy inference systems* (Mamdani & Assilian, 1975), based on FINs. Advantages of the aforementioned extensions include both a capacity to rigorously deal with nonnumeric input data and a capacity to introduce tunable nonlinearities. *Rule induction* is yet another advantage.

BACKGROUND

Lattice theory has been compiled by Birkhoff (Birkhoff, 1967). This section summarizes selected results regarding a Cartesian product lattice $(\mathsf{L},\leq)= (\mathsf{L}_1,\leq_1)\times\ldots\times(\mathsf{L}_N,\leq_N)$ of *constituent* lattices $(\mathsf{L}_i,\leq_i)$, i=1,…,N.

Given an *isomorphic* function θ_i: $(\mathsf{L}_i,\leq_i)\rightarrow(\mathsf{L}_i,\leq_i)^\partial$ in a constituent lattice $(\mathsf{L}_i,\leq_i)$, i=1,…,N, where $(\mathsf{L}_i,\leq_i)^\partial \equiv (\mathsf{L}_i,\leq_i^\partial)$ denotes the *dual* (lattice) of lattice $(\mathsf{L}_i,\leq_i)$, then an isomorphic function θ: $(\mathsf{L},\leq)\rightarrow(\mathsf{L},\leq)^\partial$ is given by $\theta(x_1,\ldots,x_N)=(\theta_1(x_1),\ldots,\theta_N(x_N))$.

Given a *positive valuation* function v_i: $(\mathsf{L}_i,\leq_i)\rightarrow\mathsf{R}$ in a constituent lattice $(\mathsf{L}_i,\leq_i)$, i=1,…,N then a positive valuation v: $(\mathsf{L},\leq)\rightarrow\mathsf{R}$ is given by $v(x_1,\ldots,x_N)=v_1(x_1)+\ldots+v_N(x_N)$.

It is well-known that a positive valuation v_i: $(\mathsf{L}_i,\leq_i)\rightarrow\mathsf{R}$ in a lattice $(\mathsf{L}_i,\leq_i)$ implies a metric function d_i: $\mathsf{L}_i\times\mathsf{L}_i\rightarrow\mathsf{R}_0^+$ given by $d_i(a,b) = v_i(a\vee b) - v_i(a\wedge b)$.

Minkowski metrics d_p: $(\mathsf{L}_1,\leq_1)\times\ldots\times(\mathsf{L}_N,\leq_N)= (\mathsf{L},\leq)\rightarrow\mathsf{R}$ are given by

$$d_p(x,y) = \left[d_1^p(x_1,y_1)+\ldots+d_N^p(x_N,y_N)\right]^{1/p}, \text{ where}$$

$x=(x_1,\ldots,x_N)$, $y=(y_1,\ldots,y_N)$, $p\in\mathsf{R}$.

An *interval* $[a,b]$ in a lattice $(\mathsf{L},\leq)$ is defined as the set $[a,b]\doteq\{x\in\mathsf{L}: a\leq x\leq b,\ a,b\in\mathsf{L}\}$. Let $\tau(\mathsf{L})$ denote the set of intervals in a lattice $(\mathsf{L},\leq)$. It turns out that $(\tau(\mathsf{L}),\leq)$ is a lattice, ordered by set inclusion.

Definition 1. The *size* Z_p: $\tau(\mathsf{L})\to\mathsf{R}_0^+$ of a lattice $(\mathsf{L},\leq)$ interval $[a,b]\in\tau(\mathsf{L})$, with respect to a positive valuation v: $(\mathsf{L},\leq)\to\mathsf{R}$, is defined as $Z_p([a,b])=d_p(a,b)$.

NEURAL/FUZZY COMPUTING BASED ON LATTICE THEORY

This section delineates modified extensions to a hierarchy of lattices stemming from the totally ordered lattice $(\mathsf{R},\leq)$ of real numbers. Then, it details the relevance of novel mathematical tools. Next, based on the previous mathematical tools, this section presents extensions of ART/SOM/FIS. Finally, it discusses comparative advantages.

Modified Extensions in a Hierarchy of Lattices

Consider the product lattice $(\Delta,\leq) = (\mathsf{R}\times\mathsf{R},\leq^{\partial}\times\leq) = (\mathsf{R}\times\mathsf{R},\geq\times\leq)$ of *generalized intervals*. A *generalized interval* (element in Δ) will be denoted by $[a,b]$ and will be called *positive* (*negative*) for $a\leq b$ ($a>b$). The set of positive (negative) generalized intervals will be denoted by Δ_+ (Δ_-) – We remark that the set of *positive* generalized intervals is isomorphic to the set of *conventional intervals* in the set R of real numbers.

A *decreasing* function θ_{R}: $\mathsf{R}\to\mathsf{R}$ is an isomorphic function θ_{R}: $(\mathsf{R},\leq)\to(\mathsf{R},\leq)^{\partial}$; furthermore, a *strictly increasing* function v_{R}: $\mathsf{R}\to\mathsf{R}$ is a positive valuation v_{R}: $(\mathsf{R},\leq)\to\mathsf{R}$. Hence, function v_Δ: $(\Delta,\leq)\to\mathsf{R}$ given by $v_\Delta([a,b])= v_{\mathsf{R}}(\theta_{\mathsf{R}}(a))+v_{\mathsf{R}}(b)$ is a positive valuation in lattice $(\Delta,\leq)$. There follows a metric function d_Δ: $\Delta\times\Delta\to\mathsf{R}_0^+$ given by $d_\Delta([a,b],[c,d])=$ $[v_{\mathsf{R}}(\theta_{\mathsf{R}}(a\wedge c))-v_{\mathsf{R}}(\theta_{\mathsf{R}}(a\vee c))] + [v_{\mathsf{R}}(b\vee d)-v_{\mathsf{R}}(b\wedge d)]$; in particular, for $\theta_{\mathsf{R}}(x)= -x$ and $v_{\mathsf{R}}(x)= x$ it follows $v_\Delta([a,b])= |a-c| + |b-d|$. Choosing *parametric* functions $\theta_{\mathsf{R}}(.)$ and $v_{\mathsf{R}}(.)$ there follow tunable nonlinearities in lattice $(\mathsf{R},\leq)$. Moreover, note that Δ is a *real linear space* with

- *addition* defined as $[a,b] + [c,d] = [a+c,b+d]$, and
- *multiplication* (by a real k) defined as $k[a,b] = [ka,kb]$.

It turns out that Δ_+ (as well as Δ_-) is *cone* in linear space Δ – Recall that a subset C of a linear space is called *cone* if for all $x\in C$ and $\lambda>0$, we have $\lambda x\in C$.

Definition 2. A *generalized interval number* (*GIN*) is a function f: $(0,1]\to\Delta$.

Let G denote the set of GINs. It follows that $(\mathsf{G},\leq)$ is a lattice, in particular $(\mathsf{G},\leq)$ is the Cartesian product of lattices $(\Delta,\leq)$. Moreover, G is a real linear space with

- *addition* defined as $(G_1 + G_2)(h) = G_1(h) + G_2(h)$, $h\in(0,1]$, and
- *multiplication* (by a real k) defined as $(kG)(h) = kG(h)$, $h\in(0,1]$.

We remark that the cardinality of set G equals $\aleph_1^{\aleph_1} =\left(2^{\aleph_0}\right)^{\aleph_1} =2^{\aleph_0\aleph_1} =2^{\aleph_1} =\aleph_2 > \aleph_1$, where $\aleph_1$ is the cardinality of the set R of real numbers.

Proposition 3. Consider metric(s) d_Δ: $\Delta\times\Delta\to\mathsf{R}_0^+$ in lattice $(\Delta,\leq)$. Let $G_1,G_2\in(\mathsf{G},\leq)$. Assuming that the following integral exists, a metric function d_{G}: $\mathsf{G}\times\mathsf{G}\to\mathsf{R}_0^+$ is given by

$$d_{\mathsf{G}}(G_1,G_2) = \int_0^1 d_\Delta(G_1(h),G_2(h))dh\,.$$

Our interest here focuses on the *sublattice* $(\mathsf{F},\leq)$ of lattice $(\mathsf{G},\leq)$, namely sublattice of *fuzzy interval numbers* (*FIN*s). A FIN is defined rigorously as follows.

Definition 4. A *fuzzy interval number* (*FIN*) F is a GIN such that either (1) both $F(h)\in\Delta_+$ and $h_1\leq h_2 \Rightarrow F(h_1)\geq F(h_2)$, for all $h\in(0,1]$ (*positive FIN*) or (2) there is a positive FIN P such that $F(h) = -P(h)$, for all $h\in(0,1]$ (*negative FIN*).

Let F_+ (F_-) denote the set of positive (negative) FINs. Note that both $\mathsf{F}_+\cup\mathsf{F}_- = \mathsf{F}$ and $\mathsf{F}_+\cap\mathsf{F}_-=\emptyset$ hold. Furthermore, F_+ (F_-) is a cone with cardinality $\aleph_1$ (Kaburlasos & Kehagias, 2006). The previous mathematical analysis may potentially produce useful techniques based on lattice vector theory (Vulikh, 1967). A *positive FIN* will simply be called "FIN". A FIN may admit different interpretations including a (fuzzy) number, an interval, and a cumulative distribution function.

Relevance of Novel Mathematical Tools

A fundamental mathematical result in *fuzzy set theory* is the "resolution identity theorem", which states that a fuzzy set can, equivalently, be represented either by its membership function or by its α-cuts (Zadeh, 1975). The aforementioned theorem has been given little attention in practice to date. However, some authors have capitalized on it by designing effective as well as efficient fuzzy inference systems (FIS) involving fuzzy numbers whose α-cuts are conventional closed intervals (Uehara & Fujise, 1993, Uehara & Hirota, 1998).

This work builds on the abovementioned mathematical result as follows. In the first place, we drop the possibilistic interpretation of a membership function. Then, we consider the corresponding "α-cuts representation". Next, we consider the metric cone F_+^N of (positive) FINs. In conclusion, we propose extensions of established neural/fuzzy algorithms, including *ART* (adaptive resonance theory), *SOM* (self-organizing map), and *FIS* (fuzzy inference systems), in F_+^N (Kaburlasos, 2007). A novelty of this work is an improved mathematical notation, which emphasizes relevance with the aforementioned "resolution identity theorem".

An Extension of Fuzzy-ARTMAP

A fuzzy-ARTMAP extension, namely *fuzzy lattice reasoning* (*FLR*), is presented in this section based on a similarity measure (function) defined in the following.

Definition 5. A *similarity measure* in a set S is a function μ: $S\times S\to(0,1]$, which satisfies the following conditions.

(S1) $\mu(a,b) = 1 \Leftrightarrow a = b$.

(S2) $\mu(a,b) = \mu(b,a)$.

(S3) $\frac{1}{\mu(a,b)}+\frac{1}{\mu(x,x)}\leq\frac{1}{\mu(a,x)}+\frac{1}{\mu(x,b)}$.

A similarity measure is defined based on a metric function next.

Proposition 6. If function d: $S\times S\to\mathsf{R}_0^+$ is a metric then function μ: $S\times S\to(0,1]$ given by $\mu(a,b) = 1/[1+d(a,b)]$ is a similarity measure.

FLR for training

FLR-0: A set $RB = \{(u_1,C_1),\ldots,(u_L,C_L)\}$ is given, where $u_l \in \mathsf{F}_+^N$ and $C_l \in \mathsf{C}$, $l=1,\ldots,L$ is a class label in the finite set C.

FLR-1: Present the next input pair $(x_i,K_i) \in \mathsf{F}_+^N \times \mathsf{C}$, i=1,…,$n$ to the initially "set" *RB*.

FLR-2: If no more pairs are "set" in *RB* then store input pair (x_i,K_i) in the *RB*; $L \leftarrow L+1$; goto step FLR-1.
Else, compute the similarity $\mu(x_i,u_l)$ of input $x_i \in \mathsf{F}_+^N$ with a "set" element $u_l \in \mathsf{F}_+^N$, $l=1,\ldots,L$ in *RB*.

FLR-3: Competition among the "set" pairs in the *RB*: Winner is pair (u_J,C_J) such that $J \doteq arg \max_{l\in\{1,\ldots,L\}} \mu(x_i,u_l)$. In case of multiple winners, choose the one with the smallest size $Z_1(.)$.

FLR-4: *Assimilation Condition*: Both (1) size $Z_1(x_i \vee u_J)$ is less than a user-defined threshold size Z_{crit}, and (2) $K_i = C_J$.

FLR-5: If the *Assimilation Condition* is not satisfied then "reset" the winner pair (u_J,C_J); goto step FLR-2.
Else, replace the winner u_J by the join-interval $x_i \vee u_J$; goto step FLR-1.

The corresponding testing phase is carried out by winner-take-all competition based on the similarity measure function $\mu(.,.)$.

An Extension of SOM

A straightforward SOM extension, namely *granular SOM* (*grSOM*), is presented in this section in cone F_+^N.

grSOM for training

GR-0: The user defines the size L of a $L\times L$ grid of neurons. Each neuron can store both a N-dimensional FIN $W_{i,j} \in \mathsf{F}_+^N$, $i,j \in \{1,\ldots,L\}$ and a class label $C_{i,j} \in \mathsf{C}$, where C is a finite set. Initially all neurons are *uncommitted*.

GR-1: Memorize the first training data pair $(x_1,K_1) \in \mathsf{F}_+^N \times \mathsf{C}$ by committing, randomly, a neuron in the $L\times L$ grid.
Repeat the following steps a user-defined number N_{epochs} of epochs.

GR-2: For each training datum $(x_k,K_k) \in \mathsf{F}_+^N \times \mathsf{C}$, k=1,…,$n$ "reset" all $L\times L$ grid neurons. Then carry out the following computations.

GR-3: Calculate the Minkowski metric distance $d_1(x_k,W_{i,j})$ between x_k and *committed* neurons $W_{i,j}$, $i,j \in \{1,\ldots,L\}$.

GR-4: Competition among the "set" (and, *committed*) neurons in the $L\times L$ grid: Winner is neuron (I,J) whose weight $W_{I,J}$ is the nearest to x_k, that is $(I,J) \doteq arg \min_{i,j\in\{1,\ldots,L\}} d_1(x_k,W_{i,j})$.

GR-5: *Assimilation Condition*: Both (1) Vector $W_{i,j}$ is in the neighborhood of vector $W_{I,J}$ on the $L\times L$ grid, and (2) $C_{I,J} = K_k$.

GR-6: If the *Assimilation Condition* is satisfied then compute a new value $W'_{i,j}$ as

$$W'_{\mathrm{i,j}} \doteq \left[1-\frac{h(k)}{1+d_1(W_{\mathrm{I,J}},W_{\mathrm{i,j}})}\right]W_{\mathrm{i,j}}+\frac{h(k)}{1+d_1(W_{\mathrm{I,J}},W_{\mathrm{i,j}})}x_{\mathrm{k}}$$

Else, “reset” the winner (I,J); goto GR-4.

GR-7: If all the $L\times L$ neurons are “reset” then commit an *uncommitted* neuron from the grid, and memorize the current training datum (x_k,K_k).
If there are no more *uncommitted* neurons then increase L by one.

The corresponding testing phase is carried out by winner-take-all competition based on the Minkowski metric $d_1(.,.)$.

An Extension of FIS

The basic idea towards novel FIS analysis and design is to employ a similarity measure function $\mu(X,A_i) = 1/[1+d(X,A_i)]$, where $X,A_i\in \mathsf{F}_+^N$, as a fuzzy membership function regarding a rule R_i: $A_i\to C_i$, where $A_i\in \mathsf{F}_+^N$, $C_i\in \mathsf{F}_+^M$, i=1,…,L (Kaburlasos & Kehagias, 2007). Advantages are presented in the following.

Comparative Advantages

First, an important advantage of the mathematical tools above is that the proposed ART/SOM/FIS extensions can handle, in any combination, numeric and/or non-numeric data, the latter include fuzzy numbers, intervals, and cumulative distribution functions.

Second, we can employ parametric decreasing (increasing) functions θ_{R}: $\mathsf{R}\to\mathsf{R}$ (v_{R}: $\mathsf{R}\to\mathsf{R}$) in a data dimension, where the function parameters can be estimated/tuned optimally towards improving performance.

Third, the proposed ART/SOM/FIS extensions can induce descriptive decision-making knowledge (i.e. rules) from the training data.

Fourth, regarding the FLR, note that a *similarity measure* function $\mu(.,.)$ can effectively replace an *inclusion measure* function $\sigma(.,.)$ – Recall that the latter (function) had replaced both of fuzzy-ARTMAP’s *Choice* (*Weber*) function and *Match* function (Kaburlasos & Petridis, 2000, Kaburlasos, Athanasiadis & Mitkas, 2007). The reason behind the aforementioned “effective” replacement is that an *inclusion measure* $\sigma(A,B)$, or $\sigma(B,A)$, considers mainly one of $A,B\in \mathsf{F}_+^N$; whereas, a *similarity measure* $\mu(A,B)$ considers both $A,B\in \mathsf{F}_+^N$ based on their corresponding metric distance.

Fifth, regarding the proposed SOM extension, note that this work carries out computations in the cone F_+ of FINs for faster data processing compared to a previous version of grSOM (Kaburlasos & Papadakis, 2006).

Sixth, regarding the proposed FIS, novel advantages include a capacity to generalize beyond a fuzzy rule’s support. The latter implies, potentially, an alleviation of the “curse of dimensionality” problem regarding the number of rules.

FUTURE TRENDS

Data-processing of FINs by multiplayer perceptrons is straightforward, as described in (Kaburlasos & Christoforidis, 2006), and it will be pursued in future work.

CONCLUSION

This article has presented novel mathematical tools for unified analysis and design of neural/fuzzy systems. We built on fuzzy set theory's "resolution identity theorem". Nevertheless, in the first place, we dropped the possibilistic interpretation of a membership function. Then, we considered the corresponding "α-cuts representation". Our interest focused on fuzzy interval numbers, or FINs for short, which can represent (fuzzy) numbers, intervals, and cumulative distribution functions. Based on lattice theory, we showed that the space of FINs is a metric cone. In conclusion, this works opens up the possibility to design FIN-to-FIN maps implementable on neural/fuzzy architectures including also tunable nonlinearities.

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TERMS AND DEFINITIONS

ART: ART stands for Adaptive Resonance Theory. That is a biologically inspired neural paradigm for, originally, clustering binary patterns. An analog pattern version of ART, namely *fuzzy-ART*, is applicable in the unit hypercube. The corresponding neural network for classification is called *fuzzy-ARTMAP*.

Dual (lattice): Given a lattice $(\mathsf{L},\leq)$, its *dual* lattice, symbolically $(\mathsf{L},\leq)^{\partial}$ or $(\mathsf{L},\leq^{\partial}) \equiv (\mathsf{L},\geq)$, is a lattice with the inverse order relation $(\geq)$.

FIS: FIS stands for Fuzzy Inference System. That is an architecture for reasoning involving fuzzy sets (typically fuzzy numbers) based of fuzzy logic.

Isomorhic (function): Given two lattices $(\mathsf{L}_1,\leq_1)$ and $(\mathsf{L}_2,\leq_2)$, an *isomorphic* function is a bijective (one-to-one) function φ: $(\mathsf{L}_1,\leq_1)\rightarrow(\mathsf{L}_2,\leq_2)$ such that $x\leq y \Leftrightarrow \varphi(x)\leq\varphi(y)$.

Lattice: A *lattice* is a poset $(\mathsf{L},\leq)$ any two of whose elements have both a greatest lower bound (g.l.b.), denoted by $x\wedge y$, and a least upper bound (l.u.b.), denoted by $x\vee y$.

Poset: A *partially ordered set* (or, *poset*, for short) is a pair $(P,\leq)$, where P is a set and $\leq$ is an order relation on P. The latter (relation) by definition satisfies (1) $x\leq x$, (2) $x\leq y$ and $y\leq x \Rightarrow x = y$, and (3) $x\leq y$ and $y\leq z \Rightarrow x\leq z$.

Positive valuation (function): Given a lattice $(\mathsf{L},\leq)$, a *positive valuation* is a function v: $(\mathsf{L},\leq)\rightarrow\mathsf{R}$, which satisfies both $v(x)+v(y) = v(x\wedge y)+v(x\vee y)$ and $x<y \Rightarrow v(x)<v(y)$.

Rule Induction: Process of learning, from cases or instances, if-then rule relationships that consist of an antecedent (if-part, defining the preconditions or coverage of the rule) and a consequent (then-part, stating a classification, prediction, or other expression of a property that holds for cases defined in the antecedent).

SOM: SOM stands for Self-Organizing Map. That is a biologically inspired neural paradigm for clustering analog patterns. SOM is often used for visualization of nonlinear relations of multi-dimensional data.

Subattice: A *sublattice* $(\mathsf{S},\leq)$ of a lattice $(\mathsf{L},\leq)$ is another lattice such that both $\mathsf{S}\subseteq\mathsf{L}$ and $x,y\in\mathsf{S} \Rightarrow x\wedge y,x\vee y\in\mathsf{S}$.